Singly Exponential Translation of Alternating Weak Büchi Automata to Unambiguous Büchi Automata

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Abstract

We introduce a method for translating an alternating weak Büchi automaton (AWA), which corresponds to a Linear Dynamic Logic (LDL) formula, to an unambiguous Büchi automaton (UBA). Our translations generalise constructions for Linear Temporal Logic (LTL), a less expressive specification language than LDL. In classical constructions, LTL formulas are first translated to alternating very weak automata (AVAs)—automata that have only singleton strongly connected components (SCCs); the AVAs are then handled by efficient disambiguation procedures. However, general AWAs can have larger SCCs, which complicates disambiguation. Currently, the only available disambiguation procedure has to go through an intermediate construction of nondeterministic Büchi automata (NBAs), which would incur an exponential blow-up of its own. We introduce a translation from general AWAs to UBAs with a singly exponential blow-up, which also immediately provides a singly exponential translation from LDL to UBAs. Interestingly, the complexity of our translation is smaller than the best known disambiguation algorithm for NBAs (broadly $(0.53n)^n$ vs. $(0.76n)^n$), while the input of our construction can be exponentially more succinct.

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1 Introduction

Automata over infinite words were first introduced by Büchi [8]. The automata used by Büchi (thus called Büchi automata) accept an infinite word if they have a run over the word that visits accepting states infinitely often. Nondeterministic Büchi automata (NBAs) are nowadays recognized as a standard tool for model checking transition systems against temporal specification languages like Linear Temporal Logic (LTL) [1,11,13,25].

NBAs belong to a larger class of automata over infinite words, also known as ω-automata. Translations between different types of ω-automata play a central role in automata theory, and many of them have gained practical importance, too. For example, researchers have started to pay attention to a kind of automata called alternating automata [19,21] in the 80s. Alternating automata not only have existential, but also universal branching. In alternating automata, the transition function no longer maps a state and a letter to a set of states, but to a positive Boolean formula over states. An alternating Büchi automaton accepts an infinite
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word if there is a run graph over the word, in which all traces visit accepting states infinitely
often. Every NBA can be seen as a special type of alternating Büchi automaton (ABA),
while the translation from ABAs to NBAs may incur an exponential blow-up in the number
of states [19]. Indeed, ABAs can be exponentially more succinct than their counterpart
NBAs [6]. Apart from their succinctness, another reason why alternating automata have
become popular in our community is their tight connection to specification logics. There
is a straightforward translation from Linear Dynamic Logic (LDL) [12, 24] to alternating
weak Büchi automata (AWAs), both recognizing exactly the \( \omega \)-regular languages. AWAs
are a special type of ABAs in which every strongly connected component (SCC) contains
either only accepting states or only rejecting states. (AWAs have also been applied to the
complementation of Büchi automata [17].) Further, there is a one-to-one mapping [5, 7, 11]
between LTL and very weak alternating Büchi automata (AVAs) [22]—special AWAs where
every SCC has only one state.

Our approach can be viewed as a generalization of earlier work on the disambiguation of
AVAs [4, 14]. The property of the very weakness has proven useful for disambiguation: to
obtain an unambiguous generalized Büchi automaton (UGBA) from an AVA, it essentially
suffices to use the nondeterministic power of the automaton to guess, in every step, the
precise set of states from which the automaton accepts. There is only one correct guess
(which provides unambiguity), and discharging the correctness of these guesses is straight
forward. AVAs with \( n \) states can therefore be disambiguated to UGBAs with \( 2^n \) states and
\( n \) accepting sets, and thus to UBAs with \( n^2 \) states.

Unfortunately, this approach does not extend easily to the disambiguation of AWAs:
while there would still be exactly one correct guess, the straightforward way to discharging
its correctness would involve a breakpoint construction [19], which is not unambiguous.

The technical contribution of this paper is to replace these breakpoint constructions by
total preorders, and showing that there is a unique correct way to choose these orders. We
provide two different reductions, one closer to the underpinning principles—and thus better
for a classroom (cf. Section 3.4)—and a more efficient approach (cf. Section 4).

Given that we track total preorders, the worst-case complexity arises when all, or almost
all, states are in the same component. To be more precise, if \( \text{tpo}(n) \) denotes the number of
total preorders on sets with \( n \) states, then our construction provides UBAs of size \( \mathcal{O}(\text{tpo}(n)) \).

As \( \text{tpo}(n) \approx \frac{n!}{2^{\Omega(n^2)\log n}} \) [3], we have that \( \lim_{n \to \infty} \sqrt[n]{\frac{\text{tpo}(n)}{n}} = \frac{1}{\sqrt{2}} \approx 0.53 \), which is a better
bound than the best known bound for Büchi disambiguation [16] (and complementation [23]),
where the latter number is \( \approx 0.76 \).

While it is not surprising that a direct construction of UBAs for AWAs is superior to a
construction that goes through nondeterminization (and thus incurs two exponential blow-ups
on the way), we did not initially expect a construction that leads to a smaller increase in

\[1\] We note that specialized model checking algorithm for Markov chains against AWAs/LDL, without
constructing UBAs, has been proposed in [9] without implementations. Nonetheless, our translation can
potentially be used as a third party tool that constructs UBA from an AWA/LDL formula for PRISM
model checker [18] without changing the underlying model checking algorithm [2].
the size when starting from an AWA compared to starting from an NBA, as AWAs can be exponentially more succinct than NBAs, but not vice versa (See [17] for a quadratic transformation).

As a final test for the quality of our construction, we briefly discuss how it behaves on AVAs, for which efficient disambiguation is available. We show that the complexity of our construction can be improved to \(n^2\) when the input is an AVA, leading to the same construction as the classic disambiguation construction for LTL/AVAs [4,14] (cf. Section 5).

We also discuss how to adjust it so that it can produce the same transition based UGBA in this case, too. The greater generality we obtain comes therefore at no additional cost.

**Related work.** Disambiguation of AVAs from LTL specifications have been studied in [4,14]. Our disambiguation algorithm can be seen as a more general form of them. The disambiguation of NBAs was considered in [15], which has a blow-up of \(O((3n)^n)\); the complexity has been later improved to \(O(n \cdot (0.7n)^n)\) in [16]. Our construction can also be used for disambiguating NBAs, by going through an intermediate construction of AWAs from NBAs; however, the intermediate procedure itself can incur a quadratic blow-up of states [14]. Nonetheless, if the input is an AWA, our construction improves the current best known approach exponentially by avoiding the alternation removal operation for AWAs [6,19].

## 2 Preliminaries

For a given set \(X\), we denote by \(B^+(X)\) the set of positive Boolean formulas over \(X\). These are the formulas obtained from elements of \(X\) by only using \(\land\) and \(\lor\), where we also allow \(\top\) and \(\bot\). We use \(\top\) and \(\bot\) to represent tautology and contradiction, respectively. For a set \(Y \subseteq X\), we say \(Y\) satisfies a formula \(\theta \in B^+(X)\), denoted as \(Y \models \theta\), if the Boolean formula \(\theta\) is evaluated to \(\top\) when we assign \(\top\) to members of \(Y\) and \(\bot\) to members of \(X \setminus Y\). For an infinite sequence \(\rho\), we denote by \(\rho[i]\) the \(i\)-th element in \(\rho\) for some \(i \geq 0\); for \(i \in \mathbb{N}\), we denote by \(\rho[i\cdots i+1] = \rho[i] \rho[i+1] \cdots\) the suffix of \(\rho\) from its \(i\)-th letter.

An alternating Büchi automaton (ABA) \(A\) is a tuple \((\Sigma, Q, \iota, \delta, F)\) where \(\Sigma\) is a finite alphabet, \(Q\) is a finite set of states, \(\iota \in Q\) is the initial state, \(\delta : Q \times \Sigma \to B^+(Q)\) is the transition function, and \(F \subseteq Q\) is the set of accepting states. ABAs allow both nondeterministic and universal transitions. The disjunctions in transition formulas model the non-deterministic choices, while conjunctions model the universal choices. The existence of both nondeterministic and universal choices can make ABAs exponentially more succinct than NBAs [6]. We assume w.l.o.g. that every ABA is complete, in the sense that there is a next state for each \(s \in Q\) and \(\sigma \in \Sigma\). Every ABA can be made complete as follows. Fix a state \(s \in Q\) and a letter \(\sigma' \in \Sigma\). If \(\delta(s, \sigma') = \bot\), we can add a sink rejecting state \(\bot\), and set \(\delta(s, \sigma') = \bot\) for every \(\sigma \in \Sigma\); If \(\delta(s, \sigma') = \top\), we can similarly add a sink accepting state \(\top\), and set \(\delta(s, \sigma') = \top\) and \(\delta(\top, \sigma) = \top\) for every \(\sigma \in \Sigma\). For a state \(s \in Q\), we denote by \(A^s\) the ABA obtained from \(A\) by setting the initial state to \(s\).

The underlying graph \(G_A\) of an ABA \(A\) is a graph \((Q, E)\), where the set of vertices is the set \(Q\) of states in \(A\) and \((q, q') \in E\) if \(q'\) appears in the formula \(\delta(q, \sigma)\) for some \(\sigma \in \Sigma\). We call a set \(C \subseteq Q\) a strongly connected component (SCC) of \(A\) if, for every pair of states \(q, q' \in C\), \(q\) and \(q'\) can reach each other in \(G_A\).

A nondeterministic Büchi automaton (NBA) \(A\) is an ABA where \(B^+(Q)\) only contains the \(\lor\) operator; we also allow multiple initial states for NBAs. We usually denote the transition function \(\delta\) of an NBA \(A\) as a function \(\delta : Q \times \Sigma \to 2^Q\) and the set of initial states as \(I\). Let \(w = w[0]w[1] \cdots \in \Sigma^\omega\) be an (infinite) word over \(\Sigma\). A run of the NBA \(A\) over \(w\) is a state sequence \(\rho = q_0q_1 \cdots \in Q^\omega\) such that \(q_0 \in I\) and, for all \(i \in \mathbb{N}\), we have that \(q_{i+1} \in \delta(q_i, w[i])\).
We denote by $\inf(\rho)$ the set of states that occur in $\rho$ infinitely often. A run $\rho$ of the NBA $A$ is accepting if $\inf(\rho) \cap F \neq \emptyset$. An NBA $A$ accepts a word $w$ if there is an accepting run $\rho$ of $A$ over $w$. An NBA $A$ is said to be unambiguous (abbreviated as UBA) [10] if $A$ has at most one accepting run for every word.

Since ABA have universal branching (or conjunctions in $\delta$), a run of an ABA is no longer an infinite sequence of states; instead, a run of an ABA $A$ over $w$ is a run directed acyclic graph (run DAG) $G_w = (V, E)$ formally defined below:

- $V \subseteq Q \times \mathbb{N}$ where $(q, 0) \in V$.
- $E = \bigcup_{\ell \geq 0} (Q \times \{\ell\}) \times (Q \times \{\ell + 1\})$ where, for every vertex $(q, \ell) \in V, \ell \geq 0$, we have that \{ $q' \in Q : (\langle q, \ell \rangle, \langle q', \ell + 1 \rangle) \in E$ \} $\models \delta(q, w[\ell])$.

A vertex $(q, \ell)$ is said to be accepting if $q \in F$. An infinite sequence $\rho = \langle q_0, 0 \rangle \langle q_1, 1 \rangle \cdots$ of vertices is called an $\omega$-branch of $G_w$ if $q_0 = i$ and for all $\ell \in \mathbb{N}$, we have $(\langle q_\ell, \ell \rangle, \langle q_{\ell + 1}, \ell + 1 \rangle) \in E$. We also say the fragment $(q_i, i)(q_{i + 1}, i + 1) \cdots$ of $\rho$ is an $\omega$-branch from $(q_i, i)$. We say a run DAG $G_w$ is accepting if all its $\omega$-branches visit accepting vertices infinitely often. An $\omega$-word $w$ is accepting if there is an accepting run DAG of $A$ over $w$.

Let $A$ be an ABA. We denote by $\mathcal{L}(A)$ the set of words accepted by $A$.

It is known that both NBAs and ABAs recognise exactly the $\omega$-regular languages. ABAs can be transformed into language-equivalent NBAs in exponential time [19]. In this work, we consider a special type of ABAs, called alternating weak Büchi automata (AWAs) where, for every SCC $C$ of an AWA $A = (\Sigma, Q, i, F)$, we have either $C \subseteq F$ or $C \cap F = \emptyset$. We note that different choices of equivalent transition formulas, e.g., $\delta(p, \sigma) = q_1$ and $\delta(p, \sigma) = q_1 \land (q_2 \lor q_3)$, will result in different SCCs. However, as long as the input ABA is weak$^2$, our proposed translation still applies.

One can transform an ABA to its equivalent AWA with only quadratic blow-up of the number of states [17]. A nice property of an AWA $A$ is that we can easily define its dual AWA $\hat{A} = (\Sigma, Q, i, F, \hat{\delta})$, which has the same statespace and the same underlying graph as $A$, as follows: for a state $q \in Q$ and $a \in \Sigma$, $\hat{\delta}(q, a)$ is defined from $\delta(q, a)$ by exchanging the occurrences of $\land$ and $\lor$ and the occurrences of $\forall$ and $\exists$ and $\hat{F} = Q \setminus F$. It follows that:

$\blacktriangleright$ Lemma 1 ([20]). Let $A$ be an AWA and $\hat{A}$ its dual AWA. For every state $q \in Q$, we have $\mathcal{L}(A^q) = \Sigma^\omega \setminus \mathcal{L}(\hat{A}^q)$.

In the remainder of the paper, we call a state of an NBA a macrostate and a run of an NBA a macrorun in order to distinguish them from those of ABA.

### 3 From AWAs to UBAs

In this section, we will present a construction of UBA $B_w$ from an AWA $A$ such that $\mathcal{L}(B_w) = \mathcal{L}(A)$. We will first introduce the construction of an NBA from an AWA given in [9] and show that this construction does not necessarily yield a UBA (Section 3.1). Nonetheless, we extract the essence of the construction and show that we can associate a unique sequence to each word (Section 3.2).

We then enrich this unique sequence with additional, similarly unique, information, which we subsequently abstract into the essence of a unique accepting macrorun of $B_w$. Developing this into a UBA whose macrorun can be uniquely mapped to the sequence (Section 3.4) is then just a simple technical exercise.

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$^2$ To make ABAs as weak as possible, one solution would be computing minimal satisfying assignments to the transition formulas, which is well defined and results in minimal possible SCCs.
3.1 From AWAs to NBAs

As shown in [19], we can obtain an equivalent NBA \( \mathcal{N}(A) \) from an ABA \( A \) with an exponential blow-up of states, which is widely known as the breakpoint construction. In [9], the authors define a different construction of NBAs \( B \) from AWAs \( A \), which can be seen as a combination of the NBAs \( \mathcal{N}(A) \) and \( \mathcal{N}(\hat{A}) \). Below we will first introduce the construction in [9] and extract its essence as a unique sequence of sets of states for each word.

The macrostate of \( B \) is encoded as a consistent tuple \((Q_1, Q_2, Q_3, Q_4)\) such that \(Q_2 = Q \setminus Q_1, Q_3 \subseteq Q_1 \setminus F\), and \(Q_4 \subseteq Q_2 \setminus \hat{F}\). The formal translation is defined as follows.

**Definition 2 ([9]).** Let \( A = (\Sigma, Q, \iota, \delta, F) \) be an AWA. We define an NBA \( B = (\Sigma, Q_B, I_B, \delta_B, F_B) \) where

- \( Q_B \) is the set of consistent tuples over \( 2^Q \times 2^Q \times 2^Q \times 2^Q \).
- \( I_B = \{ (Q_1, Q_2, Q_3, Q_4) \in Q_B \mid \iota \in Q_1 \} \).

Let \((Q_1, Q_2, Q_3, Q_4)\) be a macrostate in \( Q_B \) and \( \sigma \in \Sigma \). Then \((Q'_1, Q'_2, Q'_3, Q'_4)\) is in \( \delta_B((Q_1, Q_2, Q_3, Q_4), \sigma) \) if \( Q'_1 = \bigwedge_{s \in Q_1} \delta(s, \sigma) \) and \( Q'_2 = \bigwedge_{s \in Q_2} \hat{\delta}(s, \sigma) \) and either

- \( Q_3 = Q_4 = \emptyset \), \( Q'_3 = Q'_1 \setminus F \) and \( Q'_4 = Q'_2 \setminus \hat{F} \),
- \( Q_3 \neq \emptyset \) or \( Q_4 \neq \emptyset \), there exists \( Y_3 \subseteq Q'_1 \) such that \( Y_3 \models \bigwedge_{s \in Q_3} \delta(s, \sigma) \) and \( Q'_3 = Y_3 \setminus F \),
- and there exists \( Y_4 \subseteq Q'_2 \) such that \( Y_4 \models \bigwedge_{s \in Q_4} \hat{\delta}(s, \sigma) \) and \( Q'_4 = Y_4 \setminus \hat{F} \).

Let \( F_B = \{ (Q_1, Q_2, Q_3, Q_4) \in Q_B \mid Q_1 = Q_2 = \emptyset \} \).

Intuitively, the resulting NBA performs two breakpoint constructions: one breakpoint construction macrostate \((Q_1, Q_3)\) for \( A \) and the other breakpoint construction macrostate \((Q_2, Q_4)\) for \( \hat{A} \). Let \( w \in \Sigma^\omega \). The tuple \((Q_1, Q_3)\) in the construction uses \( Q_1 \) to keep track of the reachable states of \( A \) in a run DAG \( G_w \) over \( w \) and exploits the set \( Q_3 \) to check whether all \( \omega \)-branches end in accepting SCCs. If all \( \omega \)-branches in \( Q_3 \) have visited accepting vertices, \( Q_3 \) will fall empty, as \( Q_3 \) only contains non-accepting states. Once \( Q_3 \) becomes empty, we reset the set with \( Q'_3 = Q'_1 \setminus F \) since we need to also check the \( \omega \)-branches that newly appear in \( Q_1 \). If \( Q_3 \) becomes empty for infinitely many times, we know that every \( \omega \)-branch in \( G_w \) is accepting, i.e., all \( \omega \)-branches visit accepting vertices infinitely often. Hence \( w \) is accepted by \( A \) since there is an accepting run DAG from \( A' \). We can similarly reason about the breakpoint construction for \( \hat{A} \).

Besides that, \( \mathcal{L}(B) = \mathcal{L}(A) \), Bustan, Rubin, and Vardi [9] have also shown the following:

**Lemma 3 ([9]).** Let \( B \) be the NBA constructed as in Definition 2. Then

\[ \mathcal{L}(B) = \bigcap_{s \in S} \mathcal{L}(A) \cap \bigcap_{s \in Q_3 \setminus S} \mathcal{L}(\hat{A}) \]

where \( Q_3 \subseteq S \) and \( Q_4 \subseteq Q \setminus S \);

Let \((Q_1, Q_2, Q_3, Q_4)\) and \((Q'_1, Q'_2, Q'_3, Q'_4)\) be two macrostates of \( B \), we have that

- \( \mathcal{L}(B(Q_1, Q_2, Q_3, Q_4)) \cap \mathcal{L}(B(Q'_1, Q'_2, Q'_3, Q'_4)) = \emptyset \) if \( Q_1 \neq Q'_1 \), and
- \( \mathcal{L}(B(Q_1, Q_2, Q_3, Q_4)) = \mathcal{L}(B(Q'_1, Q'_2, Q'_3, Q'_4)) \) if \( Q_1 = Q'_1 \).

Let \( w \in \mathcal{L}(B) \) and \( \rho = (Q_1^0, Q_2^0, Q_3^0, Q_4^0)(Q_1^1, Q_2^1, Q_3^1, Q_4^1) \cdots \) be an accepting macrorun of \( B \) over \( w \). According to Lemma 3, it is easy to see that the \( Q_1 \)-set sequence \( Q_1^0Q_1^1 \cdots \) is in

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\( I_B \) is not present in [9] and we added it for the completeness of the definition.
Figure 1 An example of an AWA $\mathcal{A}$, its dual $\hat{\mathcal{A}}$ and incomplete part of the constructed $\mathcal{B}$ over $b^\omega$, where for instance the transition $\{(Q, \{q, s\}), b, (Q, \{q\})\}$ is missing.

Lemma 4. The NBA $\mathcal{B}$ defined as in Definition 2 is not necessarily unambiguous.

Proof. We prove Lemma 4 by giving an example AWA $\mathcal{A}$ for which the constructed $\mathcal{B}$ is not unambiguous. The example AWA $\mathcal{A}$ and its dual $\hat{\mathcal{A}}$ are given in Figure 1 where accepting states are depicted with double circles, initial states are marked with an incoming arrow and universal transitions are originated from a black filled circle. The transitions are by default labelled with $\Sigma = \{a, b\}$ unless explicitly labelled otherwise. We let $Q = \{p, q, s, t, r\}$. First, we can see that $b^\omega \in \mathcal{L}(\mathcal{A}^a) \cap \mathcal{L}(\mathcal{A}^b) \cap \mathcal{L}(\mathcal{A}^c) \cap \mathcal{L}(\mathcal{A}^d) \cap \mathcal{L}(\mathcal{A}^e)$. So the unique $Q_1$-sequence of all accepting macroruns in $\mathcal{B}$ over $b^\omega$ should be $Q^\omega$, according to Lemma 3. We only depict an incomplete part of $\mathcal{B}$ over $b^\omega$ where we ignore the $Q_2$ and $Q_4$ sets because we have constantly $Q_2 = \{\}$ and $Q_4 = \{\}$ by definition. One of the initial macrostates is $m_0 = (Q, \{\})$, which is also accepting. When reading the letter $b$, we always have $\{p, q, s, t, r\} = \mathcal{L}(\mathcal{B}(c, b))$. Thus, the successor of $m_0$ over $b$ is $m_1 = (Q, Q \setminus \{p, r\}) = (Q, \{q, s\})$ since the breakpoint set $Q'_5$ needs to be reset to $Q'_4 \setminus F$ when $Q_3 = \{\}$. When choosing the successor set $Q'_3$ for $Q_3 = \{q, s, t\}$ at $m_1$, we have two options, namely $\{q, s\}$ and $\{q, t\}$, since $q$ has nondeterministic choices upon reading letter $b$. Consequently, $\mathcal{B}$ can transition to either $m_2 = (Q, \{q, s\})$ or $m_3 = (Q, \{q, t\})$, upon reading $b$ in $m_1$. In fact, all the nondeterminism of $\mathcal{B}$ in Figure 1 is due to nondeterministic choices at $q$. We can continue to explore the state space of $\mathcal{B}$ according to Definition 2 and obtain the incomplete part of $\mathcal{B}$ depicted in Figure 1. Note that, we have ignored some outgoing transitions from $(Q, \{q, s\})$ since the present part already suffices to prove Lemma 4. It is easy to see that $\mathcal{B}$ has at least two accepting macroruns over $b^\omega$. Thus we have proved Lemma 4. □
3.2 Unique sequence of sets of states for each word

In the remainder of the paper, we fix an AWA $A = (\Sigma, Q, \iota, \delta, F)$. For every word $w \in \Sigma^\omega$, we define a unique sequence of sets of states associated with it as the sequence $Q_1^0 Q_1^1 Q_1^2 \cdots$ such that, for every $i \geq 0$, we have that:

1. $Q_1^i \subseteq Q$,
2. For every state $q \in Q_1^i$, $w[i \cdots] \in \mathcal{L}(A^\omega)$ and
3. For every state $q \in Q \setminus Q_1^i$, $w[i \cdots] \not\in \mathcal{L}(A^\omega)$ (or, similarly, $w[i \cdots] \in \mathcal{L}(\hat{A}^\omega)$).

These properties immediately entail the weaker local consistency requirements:

1. For every state $q \in Q_1^i$, $Q_1^{i+1} = \hat{\delta}(q, w[i])$ (entailed by P2) and
2. For every state $q \in Q \setminus Q_1^i$, $Q_1^{i+1} = \hat{\delta}(q, w[i])$ (entailed by P3).

It is obvious that, for every state $s \in Q$, $\Sigma^{\omega} = \mathcal{L}(A^\omega) \cup \mathcal{L}(\hat{A}^\omega) = \mathcal{L}(A^\omega) \cup \mathcal{L}(\hat{A}^\omega)$ holds. We define $Q_w = \{ s \in Q \mid w \in \mathcal{L}(A^\omega) \}$. This clearly provides $Q \setminus Q_w = \{ s \in Q \mid w \in \mathcal{L}(\hat{A}^\omega) \}$.

For every $w \in \Sigma^{\omega}$, we therefore have

$$w \in \bigcap_{s \in Q_w} \mathcal{L}(A^\omega) \cap \bigcap_{s \in Q \setminus Q_w} \mathcal{L}(\hat{A}^\omega) \text{ or, equivalently, } w \in \bigcap_{s \in Q_w} \mathcal{L}(A^\omega) \cap \bigcap_{s \in Q \setminus Q_w} \mathcal{L}(\hat{A}^\omega).$$

For every $i \geq 0$, P2 and P3 are then equivalent to the requirement $Q_1^i = Q_w[i \cdots]$.

To see how the local constraints L2 and L3 can be obtained from P2 and P3, respectively, we fix an integer $i \geq 0$. Let $s \in Q_1^i$, so we know that $A^s$ accepts $w[i \cdots]$. Let $S^{i+1}$ be the set of successors of $s$ in an accepting run DAG of $A^s$ over $w[i \cdots]$, i.e., $S^{i+1} = \delta(s, w[i])$. As the run DAG is accepting, we in particular have, for every $t \in S^{i+1}$, that $A^t$ accepts $w[i + 1 \cdots]$, which implies $S^{i+1} \subseteq Q_1^{i+1}$. With $S^{i+1} = \delta(s, w[i])$, this provides $Q_1^{i+1} = \delta(s, w[i])$, and thus L2.

Similarly, we can also show that, for every state $q \in Q \setminus Q_1^i$, we have $Q \setminus Q_1^{i+1} = \hat{\delta}(q, w[i])$.

As before, $\hat{A}^s$ accepts $w[i \cdots]$ for every $q \in Q \setminus Q_1^i$ by definition. We let $S^{i+1}$ be the set of successors of $q$ in an accepting run DAG of $\hat{A}^s$. This implies at the same time $S^{i+1} = \hat{\delta}(q, w[i])$ (local constraints for the run DAG and $S^{i+1} \subseteq Q \setminus Q_1^{i+1}$ (as the subgraphs starting there must be accepting). Together, this provides $Q \setminus Q_1^{i+1} = \hat{\delta}(q, w[i])$, and thus L3 also holds.

Moreover, every set $Q_1^i$ is uniquely defined based on the word $w[i \cdots]$. Therefore, the sequence $R_w = Q_1^0 Q_1^1 \cdots$ we have defined above indeed is the unique sequence satisfying P1, P2, and P3. Let us consider again the NBA construction of Definition 2: obviously, it enforces the local consistency requirements L2 and L3 on the definition of the transition relation $\delta$, which is the necessary condition for the $Q_1$-sequence being unique; the sufficient condition that $Q_1^i = Q_w[i \cdots]$ for all $i \in \mathbb{N}$ is guaranteed with the two breakpoint constructions.

In the remainder of the paper, we denote this unique sequence for a given word $w$ by $R_w$. The UBA we will construct has to guess (not only) this unique sequence correctly on the fly, but also when it leaves each SCC, as shown later.

3.3 Unique distance functions

As discussed before, we have a unique sequence $R_w = Q_1^0 Q_1^1 \cdots$ for $w$. However, as we have seen in Section 3.1, $R_w$ alone does not suffice to yield an UBA. The construction from Section 3.1, for example, validates that all rejecting SCCs can be left using breakpoints, and we have shown how that leaves leeway w.r.t. how these breakpoints are met. In this section, we discuss a different, an unambiguous (but not finite) way to check the correctness of $R_w$ by making the minimal time it takes from a state, for the given input word, to leave the rejecting SCC of $A$ or $\hat{A}$ on every branch of this run DAG. For instance, in Figure 1, it is
possible to select different successors for state \( q \) when reading a \( b \), going to either \( s \) or \( t \). One of them will lead to leaving this SCC immediately, either \( s \) (when reading a \( b \)) or \( t \) (when reading an \( a \)). For acceptance, the choice does not matter—so long as the correct choice is eventually made. On the word \( b^c \), for example in \( A \), we could go to \( t \) the first 20 times, and to \( s \) only in the \( 21^{st} \) attempt; the answer to the question 'how long does it take to leave the SCC starting in \( q \) on this run DAG?' would be 42.

The shortest time, however, is well defined. In the example automaton \( A \), it depends on the next letter: if it is \( a \), then the distance is 1 from \( t \), 2 from \( q \), and 3 from \( s \), and when it is \( b \), then the distance is 1 from \( s \), 2 from \( q \), and 3 from \( t \).

To reason about the minimal number of steps it takes from a state within a rejecting SCC that needs to leave it, we will define a distance function.

Formally, we denote by \( R \) the set of states in all rejecting SCCs of \( A \) and \( A \) the set of states in all accepting SCCs of \( A \). For a given word \( w \) and its unique sequence \( R_w \), we identify the unique distance\(^4 \) to leave a rejecting SCCs at each level \( i \) in \( G_w \) by defining a distance function \( d_i : (Q_i^1 \cap R) \cup (A \setminus Q_i^1) \rightarrow \mathbb{N}^{>0} \) for each \( i \in \mathbb{N} \).

\[ \Phi \]

**Definition 5.** Let \( w \) be a word and \( R_w = Q_0^1 Q_1^1 \cdots \) be its unique sequence of sets of states. We say \( \Phi_w = (Q_0^1, d_0)(Q_1^1, d_1) \cdots \) is consistent if, for every \( i \in \mathbb{N} \), we have \((Q_1^i, d_i) \) and \((Q_1^{i+1}, d_{i+1}) \) satisfy the following rules:

**R1.** For every state \( p \in R \cap Q_1^i \) that belongs to a rejecting SCC \( C \) in \( A \),

\[ a : (Q_1^{i+1} \setminus C) \cup \{ q \in C \cap Q_1^{i+1} | d_{i+1}(q) \leq d_i(p) - 1 \} \equiv \delta(p, w[i]) \text{ and} \]

\[ b : \text{if } d_i(p) > 1, (Q_1^{i+1} \setminus C) \cup \{ q \in C \cap Q_1^{i+1} | d_{i+1}(q) \leq d_i(p) - 2 \} \neq \delta(p, w[i]) \text{ hold.} \]

**R2.** For every state \( p \in A \setminus Q_1^i \) that belongs to an accepting SCC \( C \) in \( A \),

\[ a : (Q \setminus (Q_1^{i+1} \cup C)) \cup \{ q \in C \setminus Q_1^{i+1} | d_{i+1}(q) \leq d_i(p) - 1 \} \equiv \delta(p, w[i]) \text{ and} \]

\[ b : \text{if } d_i(q) > 1, (Q \setminus (Q_1^{i+1} \cup C)) \cup \{ q \in C \setminus Q_1^{i+1} | d_{i+1}(q) \leq d_i(p) - 2 \} \neq \delta(p, w[i]) \text{ hold.} \]

Intuitively, the distance function defines a minimal number of steps to escape from rejecting SCCs over different accepting run DAGs and maximal over different branches of one such run DAG. For instance, when \( d_i(p) = 1 \), we have that \( Q_1^{i+1} \setminus C \mid= \delta(p, w[i]) \) if \( p \in Q_1^i \cap R \), otherwise \( Q \setminus (Q_1^{i+1} \cup C) \mid= \delta(p, w[i]) \) if \( p \in A \setminus Q_1^i \). It means that \( p \) can escape from \( C \) within one step from an accepting run DAG \( G_w[\cdots] \) starting from \( \langle p, 0 \rangle \).

\[ \mathbf{Lemma 6.} \] For each \( w \in \Sigma^ \omega \), there is a unique consistent sequence \( \Phi_w = (Q_0^1, d_0)(Q_1^1, d_2) \cdots \) where \( Q_1^1 Q_1^2 \cdots \) is \( R_w \) and \( d_1, d_2 \cdots \) is the sequence of distance functions.

One can easily construct a consistent sequence of distance functions as follows. Let \( C \) be a rejecting SCC of \( A \); the case for a rejecting SCC of \( \widehat{A} \) is entirely similar. Below, we describe how to obtain a sequence of distance values for each state \( q \in C \cap Q_1^i \) with \( i \geq 0 \) in order to form a consistent sequence \( \Phi_w \). For \( q \in C \cap Q_1^i \) at the level \( i \), we first obtain an accepting run DAG \( G_w[i \cdots] \) over \( w[i \cdots] \) starting from \( \langle q, 0 \rangle \). One can define the maximal distance, say \( K \), over all branches from \( \langle q, 0 \rangle \) to escape the rejecting SCC \( C \). Such a maximal distance value must exist and be a finite value, since all branches will eventually get trapped in accepting SCCs. For all accepting run DAGs \( G'_w[i \cdots] \) over \( w[i \cdots] \) starting from the vertex \( \langle q, 0 \rangle \), there

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\(^4\) Note that, while the distance is unique, the way does not have to be. To see this, we could just expand the alphabet of \( A \) by adding a letter \( c \), and by adding \( c \) to the transitions from both \( s \) and \( t \) to \( r \). Then there are two equally short (length 2) ways from \( q \) to \( r \) whenever the next letter is \( c \).
are only finitely many run DAGs of depth \( K \) from \( \langle q, 0 \rangle \); we denote the finite set of such run
DAGs of depth \( K \) by \( P_{q,K} \). We then denote the maximal distance over one \( \textit{finite} \) run DAG
\( G_{q,i,K} \in P_{q,K} \) by \( K_{G_{q,i,K}} \). (Note that we set the distance to \( \infty \) for a finite branch in \( G_{q,i,K} \) if
it does not visit a state outside \( C \).) We then set \( d_i(q) = \min\{K_{G_{q,i,K}} : G_{q,i,K} \in P_{q,K}\} \leq K \).
One of \( G_{q,i,K} \) must provide the \( \textit{minimal} \) value, so that \( d_i(q) \) is well defined. This way, we
can define the sequence of distance functions \( d = d_0d_1 \cdots \) for the sequence \( R_w \). We can also
prove that the sequence \( R_w \times d \) is consistent by an induction on all the distance values \( k > 0 \);
We refer to Appendix A for the details.

The proof for the uniqueness of \( d \) to \( R_w \) can also be obtained by an induction on the
distance value \( k > 0 \); See also Appendix A. The intuition is that every consistent sequence of
distance functions \( c \) does not have smaller distance values than \( d \) for every state \( q \in C \cap Q^i_1 \)
(see the construction of \( d \) above), and if \( c \) does have greater distance values for some state, a
violation of the consistency constraints in Definition 5 will be found, leading to contradiction.

### 3.4 Unique total preorders

The range of the sequence \( d = d_0d_1d_2 \cdots \) of distance functions for \( R_w \) is not a priori bounded
by any given \( \textit{finite} \) number when ranging over all infinite words. Therefore, we may need
\( \textit{infinite} \) amount of memory to store \( d \). To allow for an abstraction of \( d \) that preserves
uniqueness and needs only finite memory, we will abstract the values of each function \( d_i \)
as families of total \( \textit{preorders} \), \( \{ \preceq^i_C \}_{C \in S} \), where \( S \) denotes the set of SCCs in the graph of
\( A \). For a given SCC \( C \in S \), a total preorder \( \preceq^i_C \) is a relation defined over \( H^i \times H^i \), where
\( H^i = C \cap Q^i_1 \) if \( C \subseteq R \) or \( H^i = C \setminus Q^i_1 \) if \( C \subseteq A \); As usual, \( \preceq^i_C \) is \( \textit{reflexive} \) (i.e., for each
\( q \in H^i \), \( q \preceq^i_C q \)) and \( \textit{transitive} \) (i.e., for each \( q, r, s \in H^i \), \( q \preceq^i_C r \) and \( r \preceq^i_C s \) implies \( q \preceq^i_C s \)).
We also have \( q \preceq^i_C r \) whenever \( q \preceq^i_C r \) but \( q \preceq^i_C s \). We write \( q \preceq^i_C r \) if we have \( q \preceq^i_C r \) and
\( q \preceq^i_C s \). Since \( \preceq^i_C \) is total, for every two states \( p, q \in H^i \), we have \( p \preceq^i_C q \) or \( q \preceq^i_C p \). Note
that \( \preceq^i_C \) is \( \textit{acyclic} \): it is impossible for two states \( q, p \in H^i \) satisfying \( p \preceq^i_C q \) and \( q \preceq^i_C p \).

Formally, we define a consistent sequence of total preorders as below.

\[ \textbf{Definition 7.} \ Let \ w \in \Sigma^\omega \text{ and } R_w = Q^i_1Q^i_1 \cdots \text{ be its unique sequence of sets of states. We}
\text{say } P_w = (Q^i_1, \{ \preceq^i_C \}_{C \in S})(Q^i_1, \{ \preceq^i_C \}_{C \in S}) \cdots \text{ is consistent if, for every } i \in \mathbb{N}, \text{ we have that}
(Q^i_1, \{ \preceq^i_C \}_{C \in S}) \text{ and } (Q^{i+1}_1, \{ \preceq^{i+1}_C \}_{C \in S}) \text{ satisfy the following rules:}
\]

\( \textbf{R1'.} \) \( \forall q, q' \in C \cap Q^i_1 \subseteq R \), we have that \( q \preceq^i_C q' \) iff there exists \( r \in C \cap Q^{i+1}_1 \) such that
\[ a : \{ r' \in C \cap Q^{i+1}_1 | r' \prec^{i+1}_C r \} \cup (Q^{i+1}_1 \setminus C) \models \delta(q, w[i]) \text{ and } \\
 b : \{ r' \in C \cap Q^{i+1}_1 | r' \prec^{i+1}_C r \} \cup (Q^{i+1}_1 \setminus C) \not\models \delta(q', w[i]) \text{ hold,} \]
\[ \text{where } C \subseteq R \text{ is a rejecting SCC of } A. \]

\( \textbf{R2'.} \) \( \forall q, q' \in C \setminus Q^i_1 \subseteq A \), we have \( q \preceq^i_C q' \) iff there exists \( r \in C \setminus Q^{i+1}_1 \) such that
\[ a : \{ r' \in C \setminus Q^{i+1}_1 | r' \prec^{i+1}_C r \} \cup (Q \setminus (Q^{i+1}_1 \cup C)) \models \delta(q, w[i]) \text{ and } \\
 b : \{ r' \in C \setminus Q^{i+1}_1 | r' \prec^{i+1}_C r \} \cup (Q \setminus (Q^{i+1}_1 \cup C)) \not\models \delta(q', w[i]) \text{ hold,} \]
\[ \text{where } C \subseteq A \text{ is an accepting SCC of } A. \]

As the names indicate, the Rules R1' and R2' correspond to Rules R1 and R2, respectively,
from Definition 5. We will first show that there is a consistent sequence of total preorders
for each word.

\[ \textbf{Lemma 8.} \ For each word } w \in \Sigma^\omega \text{, there exists a consistent sequence } P_w = (Q^0_0, \{ \preceq^0_C \}
_{C \in S})(Q^1_1, \{ \preceq^1_C \}_{C \in S}) \cdots \text{, where } Q^0_0Q^1_1 \cdots \text{ is the unique sequence } R_w. \]
Proof. It is simple to derive a consistent sequence $P_w = (Q_0^1, \{ \preceq_C^0 \} \in S)(Q_1^1, \{ \preceq_C^1 \} \in S) \ldots$
from $\Phi_w = (Q_0^1, d_0)(Q_1^1, d_1) \ldots$ as given in Lemma 6: We can simply select, for all $i \in \mathbb{N}$ and
$C \in S$, $\preceq_C^i$ is the total preorder over $C \cap Q_i^1$ (if $C \subseteq R$) or $C \setminus Q_1^1$ (if $C \subseteq A$) with $p \preceq_C q$
iff $d_i(p) \leq d_i(q)$. In particular, $p \preceq_C q$ iff $d_i(p) < d_i(q)$.

It is easy to verify that the sequence $P_w$ as defined above is indeed consistent. For
instance, for all $q, q' \in C \cap Q_1^1 \subseteq R$, if $q \preceq_C q'$, then $d_i(q) < d_i(q')$ by definition. Then we
can choose the $r$-state in Definition 7 (Rule R1') such that $d_{i+1}(r) = d_i(q) - 1$. (Note that
some such a state $r$ must exist since $d_i(q') > d_i(q) \geq 1$.)

Combining Definition 5 (R1) and Definition 7 (R1'), we have that Rule R1b now entails
R1'b, and Rule R1a entails R1'a, because $\{ r' \in C \cap Q_i^{i+1} \mid r' \preceq_C^{i+1} r \} \supseteq \{ r' \in C \cap Q_i^{i+1} \mid$
d_{i+1}(r') \leq d_i(q) - 1 \}$, because $d_i(q) - 1 \leq d_i(q') - 2 < d_i(q') - 1 = d_{i+1}(r)$.

The argument for accepting SCCs is using rules R2 and R2' in the same way. □

After discussing how such a sequence can be obtained, we now establish that it is unique.
Note, however, that it is unique for the correct sequence $R_w$, while there may be sequences of
total preorders that work with incorrect sequences of sets of states. (For example, a total
preorder can accommodate an infinite distance for all states, where the obligation to leave
a rejecting SCC cannot be discharged, while the local consistency constraints can be met.)
Nonetheless, a breakpoint construction ensures to obtain the unique sequence $R_w$.

Lemma 9. Let $w$ be a word in $\Sigma^*$ and $\Phi_w = (Q_0^1, d_0)(Q_1^1, d_1) \ldots$ be its unique consistent
sequence of distance functions. Let $P_w = (Q_0^1, \{ \preceq_C^0 \} \in S)(Q_1^1, \{ \preceq_C^1 \} \in S) \ldots$ be a sequence
satisfying Definition 7. Then

For every two states $q, q' \in C \cap Q_1^1 \subseteq R$, if $q \preceq_C q'$, then $d_i(q) \leq d_i(q')$, and in particular
\[ q \preceq_C q' \text{ then } d_i(q) < d_i(q'). \] (C is a rejecting SCC)

For every two states $q, q' \in C \setminus Q_1^1 \subseteq A$, if $q \preceq_C q'$, then $d_i(q) \leq d_i(q')$, and in particular
\[ q \preceq_C q' \text{ then } d_i(q) < d_i(q'). \] (C is an accepting SCC)

Proof. We only prove the first claim; the proof of the second claim is entirely similar.
Let $C$ be a rejecting SCC and $i$ be a natural number. We let $q$ and $q'$ be two states
in $C \cap Q_1^1$. In order to prove that $q \preceq_C q'$ implies $d_i(q) \leq d_i(q')$, we can just prove its
contraposition that $d_i(q') < d_i(q)$ implies $q' \preceq_C q$ for all distance values $k \geq 0$ with $d_i(q') \leq k$.
We can similarly prove that $q \preceq_C q'$ implies $d_i(q) < d_i(q')$.

Our goal is then to prove that, for all $k \geq 0$, $d_i(q') < d_i(q) \implies q' \preceq_C q$ and
\[ d_i(q') \leq d_i(q) \implies q' \preceq_C q \text{ when } d_i(q') \leq k. \] In the remainder of the proof, we will prove it
by induction over the distance value $k > 0$. Note that our claim is quantified over all natural
numbers $i$.

For the induction basis ($k = 1$), we have $d_i(q') \leq k$ by assumption. So, $d_i(q') = 1$. But
then $Q_1^{i+1} \setminus C \models \delta(q', w[i])$. Consequently, by Rule R1'b, $q'$ must be a minimal element of
$\preceq_C^i$. Hence, we have $q' \preceq_C q$. Since by assumption that $d_i(q) > d_i(q') = 1$, Rule R1 supplies
$Q_1^{i+1} \setminus C \not\models \delta(q, w[i])$. We can therefore choose $r$ from Rule R1' as a minimal element of $\preceq_C^{i+1}$
to get $S^{i+1} = \{ r' \in C \cap Q_i^{i+1} \mid r' \preceq_C^{i+1} r \} = \emptyset$. It follows that $S^{i+1} \cup (Q_1^{i+1} \setminus C) \models \delta(q', w[i])$
(R1'a) but $S^{i+1} \cup (Q_1^{i+1} \setminus C) \not\models \delta(q, w[i])$ (R1'b). By Definition 7, we have $q' \preceq_C q$. Hence,
for $k = 1$ and $d_i(q) \leq k$, it holds that $d_i(q') < d_i(q)$ implies $q' \preceq_C q$.

When $d_i(q) = d_i(q') = 1$, it directly follows that $q \not\preceq_C q'$ and $q' \not\preceq_C q$ by Definition 7,
thus also $q' \preceq_C q$ since $\preceq_C^i$ is a total preorder. Therefore, if $d_i(q') \leq d_i(q)$, then $q' \preceq_C q$,
thus also $q \preceq_C q'$ implies $d_i(q) < d_i(q')$.

For the induction step $k \rightarrow k + 1$, we have $d_i(q') = k + 1$ and we want to prove
$q' \preceq_C q$ when $k + 1 = d_i(q') < d_i(q)$, and prove $q' \preceq_C q$ when $d_i(q') = d_i(q)$ (hence
Recall that in the induction basis, we proved that \( q' \) is a minimal element with respect to \( \preceq_C \), when \( d_i(q') \leq k \). Our key observation is that, for all \( k > 0 \), all elements in \( \{ p \in C \cap Q_1^i \mid d_i(p) = k + 1 \} \) are minimal with respect to \( \preceq_C \) in the set \( \{ p \in C \cap Q_1^i \mid d_i(p) > k \} \) (See Appendix B for proof details). The intuition is that our claim is equivalent to that for every two states \( q, q' \in C \cap Q_1^i \subseteq R \), \( q \preceq_C q' \) if and only if \( d_i(q) \leq d_i(q') \) (Since \( \preceq_C \) is a preorder, we also have \( q \preceq_C q' \) iff \( d_i(q) < d_i(q') \)). Hence, the minimal elements in \( \{ p \in C \cap Q_1^i \mid d_i(p) > k \} \) (i.e., \( \{ p \in C \cap Q_1^i \mid d_i(p) = k + 1 \} \)) must also be the minimal elements with respect to \( \preceq_C \), based on our induction hypothesis.

Let \( S = \{ p \in C \cap Q_1^i \mid d_i(p) > k \} \). First, we know that \( q' \) is a minimal element with respect to \( \preceq_C \), in the set \( S \), as \( d_i(q') = k + 1 \) by assumption. Since by assumption that \( k < d_i(q') = k + 1 < d_i(q) \), we know that \( q \) is also in \( S \). Hence, \( q \preceq_C q \) holds.

We still need to prove that \( q' \preceq_C q \) under the assumption that \( d_i(q') < d_i(q) \). By assumption that \( d_i(q) > d_i(q') = k + 1 \), we pick a state \( r' \) that is minimal w.r.t. \( \preceq_C \) in the set \( \{ p \in C \cap Q_1^i \mid d_i(p) > k \} \) (and hence \( d_i(r') = k + 1 \). We then prove that the selected state \( r' \) is the r-state that witnesses \( q' \preceq_C q \) for \( R1 \) of Definition 7. The observation is that, by Definition 5, we have \( Q_1^i + 1 \cup C \cup \{ p \in C \cap Q_1^i \mid d_i(p) \leq d_i(q') - 1 \} = d_i(r' - 1) \} \not\models \delta(q', w[i]) \) but \( Q_1^i + 1 \cup C \cup \{ p \in C \cap Q_1^i \mid d_i(p) \leq d_i(r' - 1) \} = \not\models \delta(q, w[i]) \).

By induction hypothesis, for all states \( p \in C \cap Q_1^i \) such that \( d_i(p) \leq d_i(r' - 1) = k \) (i.e., \( d_i(p) < d_i(r') \)), we also have \( p \preceq_C r' \). It then follows that by Definition 7 that \( q' \preceq_C q \) holds. Hence, \( d_i(q') < d_i(q) \implies q' \preceq_C q \).

To prove that \( q \preceq_C q' \) implies \( d_i(q) < d_i(q') \), we also prove its contraposition, i.e., \( d_i(q') \leq d_i(q) \) implies \( q' \preceq_C q \) for all \( i \in \mathbb{N} \). We have already shown that \( d_i(q') < d_i(q) \) implies \( q' \preceq_C q \). Moreover, if \( d_i(q) = d_i(q') = k + 1 \), then \( q' \preceq_C q \), since both \( q' \) and \( q \) are minimal element w.r.t. \( \preceq_C \) in the set \( \{ p \in C \cap Q_1^i \mid d_i(p) > k \} \). It then follows that \( q \preceq_C q' \) implies \( d_i(q) < d_i(q') \). Hence, we have completed the proof.

By Lemma 9, for states \( p, q \in H^i \), we have both \( p \preceq_C q \iff d_i(p) = d_i(q) \) and \( p \preceq_C q \iff d_i(p) < d_i(q) \) hold for all \( i \in \mathbb{N} \), where \( H^i = C \cap Q_1^i \) if \( C \subseteq R \) and \( H^i = C \setminus Q_1^i \) if \( C \subseteq A \). Then Corollary 10 follows immediately from Lemma 6.

\[ \textbf{Corollary 10.} \text{ For each } w \in \Sigma^\omega, \text{ there is a unique consistent sequence of sets of states and total preorders } \mathcal{P}_w = (Q_1^0, \{ \preceq_C \}_{C \in S})(Q_1^1, \{ \preceq_C \}_{C \in S}) \cdots \text{ where } Q_1^0 \text{ is the unique sequence } \mathcal{R}_w. \]

In order to lift this unique set to a UBA, we need to discharge the correctness of the sequence \( Q_1^0 Q_1^1 Q_1^i \cdots \). This is, however, a relatively simple task: for the correct sequence, the total preorders provide the same rational way of creating the same accepting runs on the tails \( w[i \cdots] \) of \( w \) from the states marked as accepting in \( A \) by inclusion in \( Q_1^i \), or as accepting from \( \bar{A} \) by non-inclusion in \( Q_1^i \).

To prepare such a construction, we first define an arbitrary (but fixed) order on the SCCs of \( A \), as well as a next operator for cycling through SCCs, and fix an initial SCC \( C_0 \in S \). Recall that \( S \) is the set of all SCCs in \( A \). Note that we assume that the graph of \( A \) has at least one SCC. If this is not the case, we can simply build an unambiguous safety automaton that guesses \( \mathcal{R}_w \). Then, our construction of UBA is formalized below.

\[ \textbf{Definition 11.} \text{ Let } A = (\Sigma, Q, \iota, \delta, F) \text{ be an AWA. We define an NBA } \mathcal{B}_w = (\Sigma, Q_u, I_u, \delta_u, F_u) \text{ as follows.} \]

The macrostates of \( Q_u \) are tuples \((Q_1, Q_2, \{ \preceq_C \}_{C \in S}, S, D) \) such that
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- \( Q_1 \) and \( Q_2 \) partition \( Q \), i.e., \( Q_2 = Q \setminus Q_1 \)
- for all \( C \in S \), if \( C \subseteq R \) then \( \preceq_C \) is a total preorder over \( Q_1 \cap C \)
- for all \( C \in S \), if \( C \subseteq A \) then \( \preceq_C \) is a total preorder over \( Q_2 \cap C \)
- \( S \) is an SCC in the graph of \( A \)
- \( D \) is a downwards closed set w.r.t. the total preorder \( \preceq_S \): if \( q \in D \) then (1) \( q \in Q_1 \cap S \) if \( S \subseteq R \) resp. \( q \in Q_2 \cap S \) if \( S \subseteq A \), and (2) \( q' \preceq_S q \) implies \( q' \in D \),
- \( I_u = \{(Q_1, Q_2, \{\preceq_C\} \subseteq S, S, D) \in Q_u \mid \exists i \in Q_1, S = C_0, D = \emptyset \} \),
- Let \((Q_1, Q_2, \{\preceq_C\} \subseteq S, S, D)\) be a macrostate in \( Q_u \) and \( \sigma \in \Sigma \). Then we have that \((Q_1', Q_2', \{\preceq_C'\} \subseteq S', S', D') = \delta_u((Q_1, Q_2, \{\preceq_C\} \subseteq S, S, D), \sigma)\) if
- \( Q_1' = \sqcap_{s \in Q_1} \delta(s, \sigma) \) and \( Q_2' = \sqcap_{s \in Q_2} \delta(s, \sigma) \) (local consistency)
- for all \( C \in S \), \((Q_1, \preceq_C)\) and \((Q_1', \preceq_C')\) satisfy the requirements of Rule R1' (if \( C \subseteq R \)) resp. Rule R2' (if \( C \subseteq A \))
- if \( D = \emptyset \), then \( S' = \text{next}(S) \) and \( D' = Q_1 \cap S' \) if \( S' \subseteq R \) resp. \( D' = Q_2 \cap S' \) if \( S' \subseteq A \),
- if \( D \neq \emptyset \), then \( S' = S \) and \( D' \) is the smallest downwards closed set (see above) such that \( D' \cup (Q_1' \setminus S) = \sqcap_{s \in D} \delta(s, \sigma) \) if \( S \subseteq R \) resp. \( D' \cup (Q_2' \setminus S) = \sqcap_{s \in D} \delta(s, \sigma) \) if \( S \subseteq A \),
- \( F_u = \{(Q_1, Q_2, \{\preceq_C\} \subseteq S, S, D) \in Q_u \mid D = \emptyset \} \).

The new construction uses \( D \) as the breakpoint to ensure that the correct unique sequence \( R_w \) for each word \( w \) is obtained. The nondeterminism of the construction lies only in choosing \( Q_1' \) (which entails \( Q_2' \)) and in updating the total preorders. From an accepting macrourn of \( B_u \) over a word \( w \), one can actually construct an accepting run DAG \( G_u \) of \( A \) by selecting successors in the next level for each state \( q \) only the ones in the smallest downwards closed set \( D \) satisfying \( \delta(q, \sigma) \). This way, all branches of \( G_u \) by construction will eventually get trapped in an accepting SCC, since \( D \) will become empty infinitely often. Hence, \( L(B_u) \subseteq L(A) \). Moreover, one can construct from the unique sequence of preorders \( \Phi_w \) of a word \( w \in L(A) \) as given in Corollary 10 a unique infinite macrourn \( \rho \) of \( B_u \). Wrong guesses of the preorders for \( R_w \) will result in discontinued macrourns once a violation to R1' (or R2') has been detected. That is, there are no consistent ways to update the preorders in the next macrostate. Further, by Lemma 9, we have that \( d_i(q) = d_i(q') \Leftrightarrow q \preceq_C q' \) and \( d_i(q) < d_i(q') \Leftrightarrow q \preceq_C q' \) for all \( i \in \mathbb{N} \). So, by Definition 5 and Definition 7, one can observe that, if \( D' \neq \emptyset \), \( \sup\{d_i(q) \mid q \in D'\} = \sup\{d_i+1(q) \mid q \in D^{i+1}\} + 1 \) (choosing \( \sup\emptyset = 0 \)), where \( D' \) is the D-component of macrostate \( \rho[i] \) with \( i \in \mathbb{N} \). Since for every nonempty \( D' \), \( \sup\{d_i(q) \mid q \in D'\} \) is finite and the maximal value in \( D' \) is always decreasing, the value will eventually become 0, i.e., \( D \) always becomes empty eventually. That is, \( \rho \) must be accepting. Hence, Theorem 12 follows; See Appendix C for more details.

**Theorem 12.** Let \( B_u \) be defined as in Definition 11. Then (1) \( L(B_u) = L(A) \), (2) \( B_u \) is unambiguous.

**Example 13.** Consider again the AWW \( A \) depicted in Figure 1. Recall that, in Figure 1, the macrostate \((Q, \{q, s, t\})\) has two successors over \( b \) because of the nondeterminism in developing breakpoints. We now apply Definition 11 to construct a UBA \( B_u \) from \( A \). There are three SCCs in \( A \), namely \( C_0 = \{p\}, C_1 = \{q, s, t\} \) and \( C_2 = \{r\} \). Since \( C_0 \) and \( C_2 \) both have only one state, the total preorders for them are fixed and thus ignored here. We only need to guess the preorder over \( C_1 \). Let us consider the constructed \( B_u \) over \( b^w \) starting from the macrostate \( m_0 = (Q, \{\}, \{\preceq_C^0\}, C_1, C_1) \) where \( \preceq_C^0 \) is defined as \( \{s \preceq_C^0 q \preceq_C^0 t\} \). First, recall that \( R_{aw} = 2^Q \). Obviously, \( m_{1a} = (Q, \{\}, \{s \preceq_C^1 q \preceq_C^1 t\}, C_1, \{q, s\}) \), which corresponds to \((Q, \{q, s\})\) in Figure 1, is a valid successor of \( m_0 \) over \( b \), while \( m_{1b} = (Q, \{\}, \{s \preceq_C^1 q \preceq_C^1 t\}, C_1, \{q, t\}) \), which corresponds to \((Q, \{q, t\})\) in Figure 1, is not. The reason is that \( \{q, t\} \) is not a downwards closed set with respect to \( \preceq_C^1 \), since we have
\(s \prec_{C_1} t\) but \(s\) is missing in the breakpoint set. One may wonder whether we can change the preorder \(\preceq_{C_1}\), and consider the candidate successor \(m_{1c} = (Q, \{\}, \{q \prec_{C_1} t \prec_{C_1} s\}, \{q, t\})\). Indeed, \((q, t)\) is now a downwards closed set with respect to \(\preceq_{C_1}\). However, \((Q, \preceq_{C_1})\) and \((Q, \preceq_{C_1})\) do not satisfy the local consistency as required by Definition 7. First, we have that \(Q \setminus C_1 \cup \{\} = \delta(s, b)\). So, there do not exist \(r\)-states in \(C_1 \cap Q\) that witness \(q \prec_{C_1} s\) and \(t \prec_{C_1} s\), as required by \(R'1\) of Definition 7. In fact, one can verify that \(s \prec_{C_1} q \prec_{C_1} t\) is the only valid preorder over \(C_1\) when the input word is \(b^\omega\). This is due to the fact that when reading \(b\), the distance to escape \(C_1\) is 1 from \(s\), 2 from \(q\), and 3 from \(t\). Hence, \(m_{1c}\) must not be a valid successor of \(m_0\). The accepting macrorun of \(B_u\) (from Definition 11) over \(b^\omega\) is \((Q, \{\}, \{s \prec_{C_1} q \prec_{C_1} t\}, C_0, \{\}) \xrightarrow{b} (Q, \{\}, \{s \prec_{C_1} q \prec_{C_1} t\}, C_1, \{q, s, t\}) \xrightarrow{b} (Q, \{\}, \{s \prec_{C_1} q \prec_{C_1} t\}, C_1, \{\}) \xrightarrow{b} (Q, \{\}, \{s \prec_{C_1} q \prec_{C_1} t\}, C_2, \{\}) \xrightarrow{b} (Q, \{\}, \{s \prec_{C_1} q \prec_{C_1} t\}, C_0, \{\})\ldots.

4 Improvements and Complexity

When revisiting the construction in search for improvements, it seems wasteful to keep total preorders for all SCCs in the graph of \(A\), given that they are not interacting with each other. Can we focus on just one at a time? It proves to be possible to optimise the automaton from Definition 11 in this way, with re-establishing uniqueness proving the greatest obstacle.

The resulting automaton is smaller in practice, mainly because it only keeps track of a total preorder over only one SCC.

We provide this construction only as an improvement over the principle construction from Definition 11 for two reasons. First, while this provides quite a significant advantage where there are many small SCCs rather than one big SCC, this has little effect on the worst case (which occurs when there is one SCC, cf. Theorem 16). Second, it loosens the connection that the total preorders from Definition 11 need to be the natural abstraction of the unique distance function from Definition 5.

**Definition 14.** Let \(A = (\Sigma, Q, \iota, \delta, F)\) be an AWA. We define an NBA \(U = (\Sigma, Q_u, I_u, \delta_u, F_u)\) as follows.

- The macrostates of \(Q_u\) are tuples \((Q_1, Q_2, \preceq_C, C, D)\) such that
  - \(Q_1\) and \(Q_2\) partition \(Q\)
  - \(C\) is an SCC in the graph of \(A\) and
    - if \(C \subseteq R\) then \(\preceq_C\) is a total preorder of \(Q_1 \cap C\)
    - if \(C \subseteq A\) then \(\preceq_C\) is a total preorder of \(Q_2 \cap C\)
  - let \(M\) be the set of maximal elements of the total preorder \(\preceq_C\), and let \(H = C \cap Q_1\) if \(C \subseteq R\) resp. \(H = C \cap Q_2\) if \(C \subseteq A\); then \(D = H\) or \(D = H \setminus M\)
  - \(I_u = \{(Q_1, Q_2, \preceq_C, C, D) \in Q_u \mid \iota \in Q_1, C = C_0, D = \emptyset\}\)
  - Let \((Q_1, Q_2, \preceq_C, C, D)\) be a macrostate in \(Q_u\) and \(\sigma \in \Sigma\). Then we have that
    - \((Q_1', Q_2', \preceq'_C, C', D') \in \delta_u((Q_1, Q_2, \preceq_C, C, D), \sigma)\) if
      - \(Q_1' = \land_{s \in Q_1} \delta(s, \sigma)\) and \(Q_2' = \land_{s \in Q_2} \delta(s, \sigma)\) (local consistency)
      - if \(D = \emptyset\), then \(C' = \text{next}(C)\) and \(D' = Q_1' \cap C'\) if \(C' \subseteq R\) resp. \(D' = Q_2' \cap C'\) if \(C' \subseteq A,\)
      - if \(D = \emptyset\) then \(C' = C,\)
  - \((Q_1, \preceq_C)\) and \((Q_1', \preceq'_C)\) must satisfy the requirements of Rule \(R'1\) (if \(C \subseteq R\)) resp. \(R'2\) (if \(C \subseteq A\)) and
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* $D'$ is the smallest downward closed set w.r.t. $\preceq_{C}'$ such that $D' \cup (Q'_1 \setminus C) \models \wedge_{s \in D} \delta(s, \sigma)$ if $C \subseteq R$ resp. $D' \cup (Q'_2 \setminus C) \models \wedge_{s \in D} \delta(s, \sigma)$ if $C \subseteq A$,

$$F_u = \{ (Q_1, Q_2, \preceq_{C}, C, D) \in Q_u \mid D = \emptyset \}.$$  

The nondeterminism of the construction again lies in choosing $Q'_1$ (which entails $Q'_2$) and in updating the total preorder. One can also construct from an accepting macrorun of $\mathcal{U}$ over $w$ an accepting run DAG $G_w$ of $A$, using the same way as we did for Theorem 12. So, $L(\mathcal{U}) \subseteq L(\mathcal{A})$.

For the other direction, we first observe that the preorders of every accepting macrorun $(Q_1^0, Q_2^0, \preceq_0, S^0, D^0)(Q_1^1, Q_2^1, \preceq_1, S^1, D^1) \cdots$ of $\mathcal{U}$ over $w$ can be tightly related with the distance values of states defined in $d$. More precisely, let $D'' = D' = \emptyset$ with $i' < i$ being two consecutive accepting positions. Then for all $j \in [i', i]$, we have that:

1. for all $q \in D^j$ and all $q' \in C' \cap Q'_1$, $d_j(q) \leq d_j(q') \Leftrightarrow q \preceq_j q'$, and $d_j(q) \leq i - j$ hold,
2. for all $q \in C' \cap Q'_1$ and all $q' \in M^j = (C' \cap Q'_1) \setminus D^j$, $q \preceq_j q'$ and $d_j(q') > i - j$ hold, and
3. $m_j = \sup\{d_j(q) \mid q \in D^j\} = i - j$, using sup $\emptyset = 0$,

where $C' \subseteq R$ is a rejecting SCC of $A$. Note that $C' = C''$ for all $i' < j \leq i$. The case for $C' \subseteq A$ can be defined similarly. Let $m_j = \sup\{d_j(q) \mid q \in D^j\}$. The intuition is that all states in $M^j = (C' \cap Q'_1) \setminus D^j = \{ s \in C' \cap Q'_1 \mid d_j(s) > m_j \}$ are aggregated by construction as the maximal elements w.r.t. $\preceq_j$, while $\preceq_j$ orders all states in $D^j = \{ s \in C' \cap Q'_1 \mid d_j(s) \leq m_j \}$ exactly as in the preorders of Corollary 10. So, the correspondence between $d_j$ and $\preceq_j$ in the three items then follows naturally. For technical reasons, if $q \in D^j$ or $q' \in (C' \cap Q'_1) \setminus D^j$ do not exist in above items, we say the item above still holds. See Appendix D for proof details.

In fact, one can construct such an accepting macrorun satisfying the three items above for $\mathcal{U}$ by simulating $\mathcal{B}_u$ as follows. If $\rho = (Q_1^0, Q_2^0, \preceq_0)_{C \subseteq S}, (S^0, D^0)(Q_1^1, Q_2^1, \preceq_1)_{C \subseteq S}, S^1, D^1)\cdots$ is the accepting macrorun of $\mathcal{B}_u$ on a word $w$, then $\mathcal{U}$ has an accepting macrorun $\hat{\rho} = (Q_1^0, Q_2^0, \preceq_0, D^0)(Q_1^1, Q_2^1, \preceq_1, S^1, D^1)\cdots$ (that differs from $\rho$ only in preorders), such that

- if $S^1 \subseteq R$, then $\preceq_1$ is a total preorder on $S^1 \cap Q'_1$ where $\preceq_1 = \preceq_{S^1}'$ if $D^1 = S^1 \cap Q'_1$ and otherwise, the maximal elements $M^1$ of $\preceq_1$ are $(S^1 \cap Q'_1) \setminus D^1$, and the restriction of $\preceq_1$ to $D^1 \times D^1$ agrees with the restriction of $\preceq_{S^1}'$ to $D^1 \times D^1$, and
- similarly, if $S^1 \subseteq A$, then $\preceq_1$ is a total preorder on $S^1 \cap Q'_2$ where $\preceq_1 = \preceq_{S^1}'$ if $D^1 = S^1 \cap Q'_2$ and otherwise, the maximal elements $M^1$ of $\preceq_1$ are $(S^1 \cap Q'_2) \setminus D^1$, and the restriction of $\preceq_1$ to $D^1 \times D^1$ agrees with the restriction of $\preceq_{S^1}'$ to $D^1 \times D^1$.

It is easy to verify that $\hat{\rho}$ satisfies all local constraints for Rule R1' resp. R2'. Hence, $L(\mathcal{A}) = L(\mathcal{B}_u) \subseteq L(\mathcal{U})$, thus also $L(\mathcal{U}) = L(\mathcal{A})$.

One can show that $\hat{\rho}$ is the sole accepting macrorun of $\mathcal{U}$ over $w$ by the following facts.

(i) There is only a single initial macrostate that fits $\mathcal{B}_w$, and when we take a transition from an accepting macrostate (including the first), the next SCC is deterministically selected; (ii) Moreover, all relevant states from this SCC are in the $D^j$ component and $m_1 = \sup\{d_1(q) \mid q \in D^j\}$ is the distance to the next breakpoint (by Item (3) above), and thus the $\preceq_1$ and $D^1$ up to it. With a simple inductive argument we can thus conclude that $\hat{\rho}$ is the only such accepting macrorun. Then, Theorem 15 follows.

Note that this is a deterministic assignment that does not necessarily lead to a set $D'$ that covers all of $\preceq_{S^1}'$ or all of $\preceq_{S^1}'$ except for the maximal elements; if it does not, then this transition is disallowed.
Theorem 15. Let \( \mathcal{U} \) be defined as in Definition 14. Then (1) \( \mathcal{L}(\mathcal{U}) = \mathcal{L}(\mathcal{A}) \) and (2) \( \mathcal{U} \) is unambiguous.

We now turn to the complexity of our constructions. Let \( \text{tpo}(n) \) denote the number of total preorders over a set with \( n \) states. By [3],

\[
\text{tpo}(n) \approx \frac{n!}{\left(\frac{n}{2}\right)^{n/2}} \approx \sqrt{\frac{\pi n}{2}} \cdot \frac{1}{e} \cdot \frac{1}{\sqrt{2 \ln 2}} \approx \frac{1}{e} \approx 0.53.
\]

Thus, \( \text{tpo}(n) \approx (0.53n)^n \), which is a better bound than the best known bound \((0.76n)^n\) for Büchi disambiguation [16] and complementation [23].

Theorem 16. If \( \mathcal{A} \) has \( n \) states, then the numbers of states of \( \mathcal{U} \) and \( \mathcal{B}_u \) are \( \mathcal{O}(\text{tpo}(n)) \) and \( \mathcal{O}(n \cdot \text{tpo}(n)) \), respectively.

Proof. For both automata, the worst case occurs when all states are in the same SCC \( C \), say \( C = R \). Starting with \( \mathcal{U} \), each macrostate is a tuple \((Q_1, C \setminus Q_1, \preceq, C, D)\). There are four possibilities for the tuple, namely \( C = Q_1 = D, C = Q_1 \supseteq D, C \supseteq Q_1 = D, \) and \( C \supseteq Q_1 \supseteq D \). For each of these four cases, we can produce an injection from the tuple (macrostate) onto a total preorder \( \preceq \) over \( C \), so that we have at most \( 4 \cdot \text{tpo}(n) \) macrostates.

For \( C = Q_1 = D \), for example, we can keep the \( \preceq \) over \( C \), i.e., we set \( \preceq \equiv \preceq \). When there is strict inclusion, i.e., \( C \supseteq Q_1 \), we extend the \( \preceq \) on \( Q_1 \) to a total preorder \( \preceq' \) over \( C \) by adding the states in \( C \setminus Q_1 \) resp. \( Q_1 \setminus D \) as minimal resp. maximal elements (with their separate equivalence class). For each of the four cases, the respective mapping is injective.

As this covers all macrostates of \( \mathcal{U} \), \( \mathcal{U} \) has at most \( 4 \cdot \text{tpo}(n) \) macrostates.

For \( \mathcal{B}_u \), there are \( \mathcal{O}(n) \) possible choices for \( D \), which leads to \( \mathcal{O}(n \cdot \text{tpo}(n)) \) macrostates.

5 Discussion

The complexity of our translation is even smaller than that of the best known disambiguation algorithm for NBAs (broadly \((0.53n)^n\) vs. \((0.76n)^n\)). We can further optimise the construction of \( \mathcal{U} \) slightly by moving to transition-based acceptance conditions. Essentially, where \((Q_1', Q_2', \preceq', C, \emptyset) \in \delta_u((Q_1, Q_2, \preceq, C, D), \sigma)\), \((Q_1', Q_2', \preceq', C, \emptyset) \) would be replaced by \( \delta_u((Q_1, Q_2, \equiv, C, \emptyset), \sigma)\). (\( \equiv \) identifies all states it compares: it is the only total preorder acceptable for \( D = \emptyset \)).

This is done recursively, until the only macrostates with \( D = \emptyset \) left are those with \( Q_1 \cap R = \emptyset = Q_2 \cap A \) and (arbitrarily) \( C = C_0 \). Note that the initial macrostate has to be changed for this, too.

Removing most macrostates with \( D = \emptyset \), this reduces the statespace slightly. It is also the automaton obtained by de-generalising the standard LTL to transition-based unambiguous generalized Büchi automaton construction. We can also ‘re-generalise’: every singleton SCC can be removed from the round-robin at the cost of including an individual Büchi condition that accepts when the state \( s \) is not in \( Q_1 \) or \( Q_2 \), respectively, or if \( Q_1 \models \delta(s, \sigma) \) or \( Q_2 \models \delta(s, \sigma) \), respectively, holds. If all components are singleton, we obtain the standard construction for AVAs / LTL since the preorders of our construction given in Section 4 can be omitted. This way, the \( D \) set in a macrostate degenerates to a purely breakpoint construction.

Then, the improved complexity for AVAs matches the current known bounds \( n2^n \) for the LTL-to-UBA construction [14,25].

References

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Proof of Lemma 6

Lemma 6. For each \( w \in \Sigma^\omega \), there is a unique consistent sequence \( \Phi_w = (Q^1_0, d_0)(Q^1_1, d_2) \cdots \) where \( Q^1_0Q^1_1Q^1_2 \cdots \) is \( R_w \) and \( d_0d_1 \cdots \) is the sequence of distance functions.

Proof. Intuitively, the distance function is to define a minimal number of steps to escape from rejecting SCCs over different accepting run DAGs and maximal over different branches of one such run DAG. We first show that such a sequence of distance function exists and then prove that it is unique.

Let \( C \) be a rejecting SCC of \( A \); the proof for the case for a rejecting SCC of \( \tilde{A} \) is similar.

Below, we describe how to obtain a sequence of distance values for each state \( q \in C \cap Q^1_1 \) with \( i \geq 0 \) in order to form a consistent sequence \( \Phi_w \). For \( q \in C \cap Q^1_1 \) at the level \( i \), we first obtain an accepting run DAG \( G_w[i \cdots ] \) over \( w[i \cdots ] \) starting from \( (q, 0) \). One can define the maximal distance, say \( K \), over all branches from \( (q, 0) \) to escape the rejecting SCC \( C \). Such a maximal distance value must exist and be a finite value, since all branches will eventually get trapped in accepting SCCs. For all accepting run DAGs \( G_w[i \cdots ] \) over \( w[i \cdots ] \) starting from the vertex \( (q, 0) \), there are only finitely many run DAGs of depth \( K \) from the vertex \( (q, 0) \); we denote the finite set of such run DAGs of depth \( K \) by \( P_{q,i} \). We then denote the maximal distance \( d \) over one finite run DAG \( G_{q,i,K} \) to escape a state outside \( C \). We then set \( d(q) = \min (K_{G_{q,i,K}} : G_{q,i,K} \in P_{q,i}) \leq K \). One of \( G_{q,i,K} \) must provide the minimal value, so that \( d(q) \) is well defined. This way, we can define the sequence of distance functions \( d = d_0d_1 \cdots \) for the run DAGs \( R_w \).

We can show that the sequence \( R_w \times d \) is consistent by induction on all the distance value \( k > 0 \). We only prove the case for a state \( q \in R \cap Q^1_1 \) that belongs to a rejecting SCC in \( A \).

The proof for a state \( q \in A \setminus Q^1_1 \) is similar. We first prove the induction basis when \( k = 1 \).

Let \( q \in C \cap Q^1_1 \) be a state with \( d_1(q) \leq 1 \). By definition, we know that \( K \geq 1 \). Moreover, there must be a run DAG \( G_{q,i,K} \) of depth \( K \) as part of an accepting run DAG \( G_w[i \cdots ] \) in which the level \( 1 \) only contains the states in \( S \subseteq Q \setminus C \) such that \( S = \delta(q, w[i]) \). Since \( G_{q,i,K} \) is part of an accepting run DAG over \( w[i \cdots ] \), we also have that \( w[i + 1 \cdots ] \in \mathcal{L}(A^*) \) for all \( s \in S \). Hence, \( S \subseteq Q^1_{i+1} \), and further \( S \subseteq Q^1_{i+1} \setminus C \). It immediately follows that \( Q^1_{i+1} \setminus C = \delta(q, w[i]) \), in compliance with the rules R1a and R1b.

Now we prove the induction step (\( k \mapsto k + 1 \)). Assume that \( d_i(q) = k + 1 \) and for all distance values \( k' \leq k \), the distance function \( d = d_0d_1 \cdots \) is consistent. Again, there exists the run DAG \( G_{q,i,k+1} \) of depth \( k + 1 \) in which \( S \) is the set of states in level \( 1 \). Obviously, \( S = \delta(q, w[i]) \). Similarly, \( S \subseteq Q^1_{i+1} \) holds. For all states \( p \in S \cap C \), we have that \( d_{i+1}(p) \leq d_i(q) - 1 \) (as witnessed by the run DAG \( G_{q,i,k+1} \) over \( w[i \cdots ] \) obtained from \( G_{q,i,k+1} \) by removing level 0). Thus, we have \( S \cap C \subseteq \{ p \in C \cap Q^1_{i+1} | d_{i+1}(p) \leq d_i(q) - 1 \} \). Together with the fact that \( S \subseteq Q^1_{i+1} \setminus C \), we have that \( Q^1_{i+1} \setminus C \cup \{ p \in C \cap Q^1_{i+1} | d_{i+1}(p) \leq d_i(q) - 1 \} = \delta(q, w[i]) \), which is consistent with the rules R1a.

We can prove R1b easily by contraposition. Assume that \( Q^1_{i+1} \setminus C \cup \{ p \in C \cap Q^1_{i+1} | d_{i+1}(p) \leq d_i(q) - 2 \} = \delta(q, w[i]) \). Then, there exists a run DAG \( G_{q,i,K} \) in which the level \( 1 \) contains all the states in \( Q^1_{i+1} \setminus C \cup \{ p \in C \cap Q^1_{i+1} | d_{i+1}(p) \leq d_i(q) - 2 \} \). Since \( d \) is consistent when the distance value is not greater than \( K \), so \( K_{G_{q,i,K}} \leq K \) by induction hypothesis. Thus, by definition, we should have \( d_i(q) = k \), leading to contradiction.

Therefore, \( R_w \times d \) is a consistent sequence.

Now we prove that the distance function \( d \) is unique to \( R_w \). We observe that a consistent sequence \( c = c_1c_2 \cdots \) will provide an accepting run DAG for all tails \( w[i \cdots ] \) with \( i \in \mathbb{N} \) by always choosing the satisfying sets from R1a and R2a, respectively, from a state \( q \) in the
domain of $c_i$, we will leave its SCC $C$ from level $i$ of the run DAG in $c_i(q)$ steps, so that no run can get stuck in a rejecting SCC.

This also provides $c_i(q) \geq d_i(q)$ for all $i \in \mathbb{N}$ and all $q$ in their domain, by definition of $d_i$.

We now show by induction that, for all $k > 0$ and all $i \in \mathbb{N}$, the pre-image of $c_i$ and $d_i$ for $k$ coincide.

The induction basis is the case of $k = 1$, and thus $d_i(q) = 1$. For this to happen, it requires that $C$ can be left immediately, which would then allow for using rule R1b or R2b, as the left set of the union alone suffices for satisfaction. $c_i(q) = 1$ is therefore the only possible assignment (and in compliance with rules R1a and R1b).

The induction step is from $k$ to $k + 1$.

Let $d_i(q) = k + 1$. We have already shown $c_i(q) \geq k + 1$.

By definition, there is an accepting run DAG from $q$ at level $i$, such that $C$ is left in $k + 1$ steps. We fix such a run DAG. We observe that for every successor $s$ in level $i + 1$ we have that it is either outside of $C$, or $d_{i+1}(s) \leq k$. Using the induction hypothesis, the latter entails $c_{i+1}(s) \leq k$. Therefore, rule R1a or R2a applies. We now assume for contradiction that the respective rule R1b or R2b does not apply. But then we can satisfy $\delta(p, w[i])$ or $\delta(q, w[i])$, respectively, by only those states not in $C$ or with $d_{i+1}(s) < k$, which would entail $d_i(q) \leq k$ (by making such a choice and inserting the witnessing run graphs in level $i + 1$). This closes the contradiction, and provides $c_i(q) = k + 1$.

This completes the induction and provides the Lemma.

\section*{B Proof of Lemma 9}

\begin{lemma}
Let $w$ be a word in $\Sigma^*$ and $\Phi_w = (Q_0, d_0)(Q_1, d_1) \cdots$ be its unique consistent sequence of distance functions. Let $P_w = (Q_0, \{\leq_C^i\} \subseteq S)(Q_1, \{\leq_C^i\} \subseteq S) \cdots$ be a sequence satisfying Definition 7. Then

\begin{itemize}
  \item For every two states $q, q' \in C \cap Q_i^k \subseteq R$, if $q \leq_C^i q'$, then $d_i(q) \leq d_i(q')$, and in particular if $q \prec_C^i q'$, then $d_i(q) < d_i(q')$. (C is a rejecting SCC)
  \item For every two states $q, q' \in C \setminus Q_i^k \subseteq A$, if $q \leq_C^i q'$, then $d_i(q) \leq d_i(q')$, and in particular if $q \prec_C^i q'$, then $d_i(q) < d_i(q')$. (C is an accepting SCC)
\end{itemize}
\end{lemma}

\begin{proof}
We only prove the first claim; the proof of the second claim is entirely similar.

Let $C$ be a rejecting SCC and $i$ be a natural number.

First, in order to prove that $q \leq_C^i q'$ implies $d_i(q) \leq d_i(q')$, we can just prove its contraposition that $d_i(q') < d_i(q)$ implies $q' \prec_C^i q$ for all distance values $k \geq 1$ with $d_i(q') \leq k$. We can prove $q \prec_C^i q'$ implies $d_i(q) < d_i(q')$ similarly.

In the remainder of the proof, we will prove the claim by an induction over distance value $k > 0$ and assume that $d_i(q') \leq k$. Our goal is to prove that $d_i(q') < d_i(q) \implies q' \prec_C^i q$ and $d_i(q') \leq d_i(q) \implies q' \leq_C^i q$. Note that the claim is quantified over all natural number $i$.

For the induction basis ($k = 1$), we have $d_i(q') \leq k$ by assumption. So $d_i(q') = 1$. But then $Q_1^i \cup C \models \delta(q', w[i])$. Consequently, by Rule R1b, $q'$ must be a minimal element of $\leq_C^i$, and we have $q' \leq_C^i q$. Since by assumption that $d_i(q') > d_i(q) = 1$, Rule R1 supplies $Q_1^{i+1} \cap C \models \delta(q, w[i])$. We can therefore choose $r$ from Rule R1' as a minimal element of $\leq_C^{i+1}$ to get $S_1^{i+1} = \{ r' \in C \cap Q_1^{i+1} \mid r' \leq_C^{i+1} r \} = \emptyset$. It follows that $S_1^{i+1} \cup (Q_1^{i+1} \setminus C) \models \delta(q, w[i])$ (R1'a) but $S_1^{i+1} \cup (Q_1^{i+1} \setminus C) \not\models \delta(q, w[i])$ (R1'b). By Definition 7, we have $q' \prec_C^i q$. Hence, for $k \in \mathbb{N}$ with $d_i(q') \leq k = 1$, it directly follows that $q \not\prec_C^i q'$ and $q' \not\prec_C^i q$ by Definition 7, thus also

\end{proof}
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$q' \succeq_C q$ since $\succeq_C$ is a total preorder. Therefore, if $d_i(q') \leq d_i(q)$, then $q' \succeq_C q$, thus also $q' \preceq_C q'$ implies $d_i(q') < d_i(q)$. 

For the induction step $k \mapsto k + 1$, we have $d_i(q') = k + 1$ and we want to prove $q' \preceq q$ when $k + 1 = d_i(q') < d_i(q)$, and prove $q' \preceq_C q$ when $d_i(q') = d_i(q)$ (hence $d_i(q') \leq d_i(q) \implies q' \succeq_C q$).

First, there must be a state $s \in C \cap Q_1$ with $d_i(s) = k$ according to R1 in Definition 5; we then pick such a state $s$. By induction hypothesis, for all $p, p' \in C \cap Q_1 \wedge d_i(p) < k \land d_i(p') > d_i(p)$, we have that $p \preceq_C p'$. Moreover, our claim is equivalent to that for every two states $q, q' \in C \cap Q_1 \subseteq R, q \preceq_C q'$ if and only if $d_i(q) \leq d_i(q')$ (Since $\preceq_C$ is a preorder, we also have $q \preceq_C q'$ iff $d_i(q) < d_i(q')$). Also, the claim has been proved for all $i \in \mathbb{N}$ in the induction basis. Therefore, the following holds for every $s' \in C \cap Q_1$: 

(1) $s' \preceq_C s' \iff d_i(s') \leq k = d_i(s)$, and 
(2) $s' \preceq_C s \iff d_i(s') < k = d_i(s)$.

Item (1) implies that $\{s' \in C \cap Q_1 \mid s' \preceq_C s\} = \{s' \in C \cap Q_1 \mid d_i(s') \leq d_i(s) = k\}$, while Item (2) gives that $\{s' \in C \cap Q_1 \mid s' \preceq_C s\} = \{s' \in C \cap Q_1 \mid d_i(s') \leq k - 1 = d_i(s) - 1\}$. Hence, by Definitions 5 and 7, we have that, for state $p \in C \cap Q_1$, $d_i(p) \leq k$ iff $(Q_i^{+ \bot} \setminus C) \cup \{s' \in C \cap Q_1 \mid d_i(s') \leq d_i(p) - 1 \leq k - 1\} \models (p, w[i](\{i\})$ (by R1 of Definition 5) iff $(Q_i^{+ \bot} \setminus C) \cup \{s' \in C \cap Q_1 \mid s' \preceq_C s\} \models (p, w[i])$ (by Item (2)). Below we only explain why the transition relation in Definition 5 holds when $d_i(p) < k$. According to Definition 5, we have $(Q_i^{+ \bot} \setminus C) \cup \{s' \in C \cap Q_1 \mid d_i(s') \leq d_i(p) - 1 \models (p, w[i])$. So, if $d_i(p) < k$, then $\{s' \in C \cap Q_1 \mid d_i(s') \leq d_i(p) - 1 \} \subseteq \{s' \in C \cap Q_1 \mid d_i(s') \leq k - 1\}$. It follows that $(Q_i^{+ \bot} \setminus C) \cup \{s' \in C \cap Q_1 \mid d_i(s') \leq k - 1\} \models (p, w[i])$. Note that 

$d_i(s') \leq d_i(p) - 1 \leq k - 1$ above is actually $d_i(s') \leq k - 1$.

Further, by applying R1 in Definition 5, it follows that, for state $p \in C \cap Q_1$, $d_i(p) = k + 1$ iff $(Q_i^{+ \bot} \setminus C) \cup \{s' \in C \cap Q_1 \mid d_i(s') = d_i(p) - 1 = k = d_i(s)\} \models (p, w[i])$ (by R1a of Definition 5) iff $(Q_i^{+ \bot} \setminus C) \cup \{s' \in C \cap Q_1 \mid s' \preceq_C s\} \models (p, w[i])$ (by Item (1), and we denote it as E1). With this, Rule R1' implies that $p \in C \cap Q_1$ with $d_i(p) = k + 1$ must be a minimal element w.r.t. $\preceq_C$ in the set $\{p \in C \cap Q_1 \mid d_i(p) > k\}$; otherwise, there must be some $p \in C \cap Q_1$ with $p \preceq_C q'$ and $d_i(p) \geq k + 1 = d_i(q') > k$, and an element $r \in C \cap Q_1$ satisfying R1'a and R1'b for $p$ and $q'$, violating the induction hypothesis and Definition 5.

Since we have $q' \in C \cap Q_1 \wedge d_i(q') = k + 1$ by assumption, $q'$ is also a minimal element w.r.t. $\preceq_C$ in the set $\{p \in C \cap Q_1 \mid d_i(p) > k\}$. Let $S = \{p \in C \cap Q_1 \mid d_i(p) > k\}$. First, we already proved that $q'$ is a minimal element w.r.t. $\preceq_C$ in the set $S$. Since by assumption that $k < d_i(q') = k + 1 < d_i(q)$, we know that $q$ is also in $S$. Hence, $q' \preceq_C q$ holds since $d_i(q') < d_i(q)$. Moreover, by assumption that $d_i(q) > d_i(q') = k + 1$, then we pick a state $r$ that is minimal w.r.t. $\preceq_C$ in the set $\{p \in C \cap Q_1 \mid d_i(p) > k\}$. Recall that by induction hypothesis, for all $d_i(q') \leq k$, we have that $q' \preceq_C q$ iff $d_i(q') < d_i(q)$ for all $i \in \mathbb{N}$. Hence, $r \in \{p \in C \cap Q_1 \mid d_i(p) > k\} = \{p \in C \cap Q_1 \mid d_i(p) > k, s \preceq_C p\}$. Together with $p \preceq_C \{s \in C \cap Q_1 \mid s \preceq_C p\}$ (E1), $Q_1^{+ \bot} \setminus C \cup \{p \in C \cap Q_1 \mid p \preceq_C r\} \models (q', w[i])$ (R1a). Moreover, $Q_1^{+ \bot} \setminus C \cup \{p \in C \cap Q_1 \mid p \preceq_C r\} \models (q', w[i])$ (R1b) since by induction hypothesis, it is equivalent to $Q_1^{+ \bot} \setminus C \cup \{p \in C \cap Q_1 \mid d_i(p) \leq d_i(r) - 1 = k \leq d_i(q) - 2 \models (q', w[i])$ (see Definition 5).

Then R1' implies $q' \preceq_C q$. Hence, we also have that $d_i(q') < d_i(q)$ implies that $q' \preceq_C q$.

To prove that $q' \preceq_C q$ implies $d_i(q') < d_i(q)$, we also prove its contraposition, i.e., $d_i(q') \leq d_i(q)$ implies $q' \preceq_C q$ for all $i \in \mathbb{N}$. We have already shown that $d_i(q') < d_i(q)$ implies $q' \preceq_C q$. Moreover, if $d_i(q') = d_i(q) = k + 1$, then $q' \preceq_C q$, since both $q'$ and $q$ are minimal element w.r.t. $\preceq_C$ in the set $\{p \in C \cap Q_1 \mid d_i(p) > k\}$. It then follows that $q' \preceq_C q'$ implies $d_i(q') < d_i(q')$. Hence, we have completed the proof. △
Proof of Theorem 12

**Theorem 12.** Let $B_u$ be defined as in Definition 11. Then (1) $\mathcal{L}(B_u) = \mathcal{L}(A)$, (2) $B_u$ is unambiguous.

**Proof.** We first observe that, for an accepting macrorun $\rho = (Q_1^0, Q_2^0, (\preceq_C^0)_{C \in S}, S^0, D^0)\text{ (w.r.t. } \preceq_C^0)$ downward closed set $Q' \subseteq C \cap Q_1^0$ such that $Q' \cup (Q_1^0 \setminus C) = \delta(q', w[j])$).

We now show that all the branches in the run DAG cannot get stuck in $C$. As the macrorun $\rho$ is accepting, there must be a next time $k > j$ where either $Q_k^0 \cap C = \emptyset$ (which trivially means that the branches do not get stuck in $C$) or $D^k = Q_k^0 \cap C$—either happens at the latest after $|S|$ accepting macrostates have been visited. Recall that in the construction, when $D^{k-1} = \emptyset$ (and thus a visit to accepting macrostate) and $C = S_k = \text{next}(S^{k-1})$, we have that $D^k = Q_k^0 \cap C$ as $C \subseteq R$ is a non-accepting SCC. For the latter case, it is to show by induction that all branches in the run DAG originating from $q$ are henceforth either not in $C$, or in $D$, so that $C$ is left at the latest when the $|S| + 1$st accepting macrostate is visited.

The reason why the branches are in $D$ is that according to the construction, we only leave the smallest downward closed set of successors for $D$ in $D'$. Since $\rho$ visits empty $D$-sets for infinitely many times, the run DAG must not be stuck in $C$ for all $C \in S$. Therefore, the run DAG $G_{q,i}$ is accepting. It follows that $w[i \cdots] \in \mathcal{L}(A^0)$.

The proof for (2) is similar.

Using this, we first obtain $\mathcal{L}(B_u) \subseteq \mathcal{L}(A)$ (as an accepting macrorun must satisfy $i \in Q_1^0$).

Second, it implies that $R_w = Q_0^0 Q_1^0 Q_1^2 \cdots$ holds for all accepting macroruns.

With Corollary 10 and the observation that the update of the last two components of a macrostate ($S$ and $D$) are deterministic, this entails unambiguity (there is at most one accepting macrorun). Note that any wrong guesses for preorders will violate the local consistency rules and those macroruns will therefore discontinue the moment violations are found.

Finally, if $w \in \mathcal{L}(A)$, then we have for the unique sequence $R_w = Q_0^0 Q_1^0 Q_1^2 \cdots$ that $\tau \in Q_1^0$

and we can use Lemma 6 to construct the corresponding unique distance functions $d_0 d_1 d_2 \cdots$. Now we show how to construct an accepting macrorun $\rho = (Q_1^0, Q_2^0, (\preceq_C^0)_{C \in S}, S^0, D^0)$ of $B_u$ over $w$ where $Q_2^0 = Q \setminus Q_1^0$ and $Q_1^0 Q_2^1 Q_3^1 \cdots$ is of course the unique sequence $R_w$. We will set the preorders $\{\preceq_C \}_{C \in S}$ as defined in Lemma 8, based on the distance functions $d_0 d_1 d_2 \cdots$. The updates of $S$ and $D$ are then deterministic with respect to $Q_1$ and $\{\preceq_C \}_{C \in S}$. Apparently, the preorders meet all the local consistency constraints, according to Lemma 8. So, the macrorun $\rho$ is of infinite length.

By Lemma 9 and Corollary 10, we also know that it is the unique preorder sequence for $R_w$, which gives $q \preceq_C q' \iff d_i(q) \leq d_i(q')$ and $q \preceq_C q' \iff d_i(q) < d_i(q')$ for all $i \in \mathbb{N}$. With this, by Definition 5 and Definition 7, it is now easy to show with an inductive argument similar to the one in Lemma 9 that, if $D^{i} \neq \emptyset$, $\sup\{d_i(q) \mid q \in D^{i}\} = \sup\{d_{i+1}(q) \mid q \in D^{i+1}\} + 1$ (choosing $\sup\emptyset = 0$). Since all the distance values of the states in $D^{i} \neq \emptyset$ are finite and the maximal value in $D^{i}$ is decreasing, the value will eventually become 0. In other words, for every $i > 0$ with $D^{i} \neq \emptyset$, there will be some $j > i$ such that $D^{j} = \emptyset$. Thus, the macrorun $\rho$ must be accepting.
We therefore also have \( \mathcal{L}(A) \subseteq \mathcal{L}(B_u) \).

D Proof of Theorem 15

**Proof.** We first observe that, for an accepting macrorun \( \rho = (Q_0, Q_0^1, \preceq_0, C^0, D^0) \)
\((Q_1, Q_2^1, \preceq_1, C^1, D^1)\) \((Q_1^2, Q_2^2, \preceq_2, C^2, D^2) \cdots \) on a word \( w \) we have that

1. \( q \in Q_1 \) implies \( w[i \cdots] \in A^0 \) and
2. \( q \in Q_2 \) implies \( w[i \cdots] \in A^1 \),

with exactly the same proof as in Theorem 12.

Using this, we first similarly obtain \( \mathcal{L}(U) \subseteq \mathcal{L}(A) \) (as an accepting macrorun must satisfy
\( \iota \in Q_i^\dagger \) and that \( B_w = Q_i^\dagger Q_i^2 \cdots \) holds for all accepting macroruns.

Next we show that \( U \) can simulate \( B_u \): if \( \rho = (Q_0, Q_0^1, \preceq_0, C^0, D^0) \)
\((Q_1, Q_2^1, \preceq_1, C^1, D^1)\) \((Q_1^2, Q_2^2, \preceq_2, C^2, D^2) \cdots \) is an accepting macrorun of \( B_u \) on a word \( w \),
then \( U \) has an accepting macrorun \( \hat{\rho} = (Q_0, Q_0^1, \preceq_0, C^0, D^0) \)
\((Q_1, Q_2^1, \preceq_1, C^1, D^1)\) \((Q_1^2, Q_2^2, \preceq_2, C^2, D^2) \cdots \), where

- if \( S_i \subseteq R \), then \( \preceq_i \) is a total preorder on \( S_i \cap Q_1 \)
- is \( \preceq_i = \preceq_i^s \) if \( D^i = S_i \cap Q_1^i \) and
- otherwise, the maximal elements of \( \preceq_i \) are set to \( (S_i \cap Q_1^i) \setminus D^i \), and the restriction of
  \( \preceq_i \) to \( D^i \times D^i \) agrees with the restriction of \( \preceq_i^s \) to \( D^i \times D^i \), and

- similarly, if \( S_i \subseteq A \), then \( \preceq_i \) is a total preorder on \( S_i \cap Q_2 \)
- is \( \preceq_i = \preceq_i^s \) if \( D^i = S_i \cap Q_2^i \) and
- otherwise, the maximal elements of \( \preceq_i \) are set to \( (S_i \cap Q_2^i) \setminus D^i \), and the restriction of
  \( \preceq_i \) to \( D^i \times D^i \) agrees with the restriction of \( \preceq_i^s \) to \( D^i \times D^i \).

Let \( m_i = \max\{d_i(q) \mid q \in D^i\} \). Intuitively, the total preorder \( \preceq_i \) simply orders those
states in \( s \in S_i \cap Q_1 \) resp. \( s \in S_i \cap Q_2 \) with \( d_i(s) \leq m_i \) correctly, while aggregating all such
states \( s \) with \( d_i(s) > m_i \) as maximal elements. It is easy to extend the proof of Theorem 12
to show that this satisfies all local constraints for Rule R1’ resp. R2’. Note that our preorders
\( \preceq_i \) are no longer defined over all SCCs, so Lemma 9 may not entirely hold here. Now
we show that \( (Q_1^i+1, Q_2^i+1, \preceq_{i+1}, S_{i+1}, D^{i+1}) \) is a valid \( w[i+1] \)-successor of \( (Q_1^i, Q_2^i, \preceq_i, S_i, D^i) \).
First, the local consistency for the reachable states \( Q_1^i+1 \) and \( Q_2^i+1 \) clearly holds since \( \rho \)
also visits the same set of reachable states. If \( D^i = \emptyset \), two constructions behave the same.
So we only need to show it is valid when \( D^i \neq \emptyset \). We next show that the requirements of
Rule R1’ are met; the proof for Rule R2’ is similar. If \( D^i = S_i \cap Q_1 \), then \( D^{i+1} \) is the
smallest downward closed set w.r.t. \( \preceq_{i+1} \) such that \( D^{i+1} \cup (Q_1^i+1 \setminus S_{i+1}) \models \land_{e \in D_i} \delta(s, \sigma) \). If
\( D^i+1 = S_i+1 \cap Q_1^i+1 \), then \( \preceq_i+1 = \preceq_{i+1} \) and the consistency clearly holds. If \( D^{i+1} \subset S_i+1 \cap Q_1^i+1 \),
then, for every pair of states \( q, q’ \) with \( q \preceq_{i+1} q’ \), there must be a state \( r \in D^{i+1} \) satisfying
Definition 7, since \( D^{i+1} \cup (Q_1^i+1 \setminus S_{i+1}) \models \land_{e \in D_i} \delta(s, \sigma) \) and \( \preceq_i+1 = \preceq_{i+1} \) over \( D^{i+1} \times D^{i+1} \),
where \( S_{i+1} = S_i \). If \( D^i \neq S_i \cap Q_1 \), then we have \( D^i \subset S_i \cap Q_i \). For states \( q, q’ \in D^i \) with
\( q \preceq_i q’ \), the proof is similar. Consider \( q \in D^i, q’ \in (S_i \cap Q_1^i) \setminus D^i \) with \( q \preceq_i q’ \); it is impossible
that \( D^{i+1} = S_i+1 \cap Q_1^i+1 \). This is because that since \( \rho[i] \) and \( \rho[i+1] \) are consistent sequence,
they \( D^i \) will include all states from \( S_i \subset Q_1 \), violating the assumption and Definition 7.
So, it must be the case that \( D^{i+1} \subset S_i+1 \cap Q_1^i+1 \). Then we can just select the \( r \)-state of
Definition 7 as a minimal element \( (S_i+1 \cap Q_1^i+1) \setminus D^{i+1} \), satisfying R1’ in Definition 7. Hence,
the macrorun \( \hat{\rho} \) is infinite and visits infinitely many empty \( D \)-sets.

This provides \( \mathcal{L}(U) \supseteq \mathcal{L}(B_u) \). With \( \mathcal{L}(B_u) = \mathcal{L}(A) \) (Theorem 12), we now have \( \mathcal{L}(U) = \mathcal{L}(A) \).
To show that there is only one accepting macrorun, we turn the argument of assigning values around. Our proof idea is to establish some properties of every accepting macrorun in \( \mathcal{U} \) and prove that there is only one macrorun satisfying such properties.

We have already established that \( \mathcal{R}_w = Q_0^1, q_1^1, Q_2^1, \ldots \) holds, and will use the unique extension \( \Phi_w = (Q_0^1, d_0)(Q_1^1, d_1) \cdots \) to distance functions (Lemma 6).

Let \( \hat{\rho} = (Q_0^1, Q_2^1, \ldots, 0, C^1, D^1)(Q_1^1, Q_2^1, \ldots, 1, C^1, D^1)(Q_1^2, Q_2^2, \ldots, 2, C^2, D^2) \cdots \) be an accepting macrorun of \( \mathcal{U} \) on a word \( w \). Let \( i > 0 \) be an accepting macrostate position in \( \hat{\rho} \), and let \( i' < i \) be the last accepting macrostate position that occurred before \( i \).

We assume \( C^i \subseteq R \), the case \( C^i \subseteq A \) is entirely similar. We note that \( C^i = C^0 \) for all \( i' < j \leq i \). Hence, in the following, we actually work on the SCC \( C^i \).

We now show by induction over \( j \) that, for all \( i' < j \leq i \) we have that, for \( m_j = \operatorname{sup}(d_j(q) \mid q \in D^j) \), the total preorder \( \preceq_j \) simply orders those states in \( C^i \cap Q_1^1 \) correctly, while the remaining states are maximal elements of \( \preceq_j \):

1. for all \( q \in D^j \) and all \( q' \in C^i \cap Q_1^1 \), \( d_j(q) \leq d_j(q') \Leftrightarrow q \preceq_j q' \), and \( d_j(q) \leq i - j \) hold,
2. for all \( q \in C^i \cap Q_1^1 \) and all \( q' \in (C^i \cap Q_1^1) \setminus D^j \), \( q \preceq_j q' \) and \( d_j(q') > i - j \) hold, and
3. \( m_j = \operatorname{sup}(d_j(q) \mid q \in D^j) = i - j \), using \( \operatorname{sup} \emptyset = 0 \).

For the induction basis, this is true by definition for \( j = i \). By assumption, \( D^j = \emptyset \) and by Definition 11, \( \preceq_j \) identifies only one equivalence class—the maximal equivalence class, since \( D^j \) is the smallest downward closed set w.r.t. \( \preceq_j \) such that \( D^j \cup (Q_1^1 \setminus C^0) \models \bigwedge_{s \in D^j} \delta(s, w[i]) \).

By definition, either \( D^j = Q_1^1 \cap C^0 \) or \( D^j = (Q_1^1 \cap C^0) \setminus M^j \) holds where \( M^j \) is the maximal elements of \( \preceq_j \). Hence, for all \( q \in C^i \cap Q_1^1 \), \( q' \in (C^i \cap Q_1^1) \setminus D^j \), \( q \preceq_j q' \) and \( d_j(q') > 0 \) holds always. Therefore, Item (2) holds. Moreover, Item (3) clearly holds. For Item (1), since \( q \) does not exist, we simply say Item (1) is true for technical reason.

For the induction step \( j 
- j - 1 \) (assuming \( j > j' + 1 \)), Rules R1 and R1’ imply with (1) and (2) hold also for \( j - 1 \).

For \( j = i \), and thus \( D^j = \emptyset \), by Rule R1 it is exactly those states with \( d_j(q) = 1 \) that are in \( D^{j-1} \). Assume that \( D^{j-1} \neq \emptyset \). Clearly, for all \( q \in D^{j-1}, d_{j-1}(q) = 1 \leq i - (j - 1) = 1 \).

Hence, Item (3) holds. For \( q' \in D^{j-1} \subseteq C^i \cap Q_1^{i-1} \), we have \( d_{j-1}(q) = d_{j-2}(q') \) hold. Further, \( q \preceq_{j-1} q' \) holds since if there is \( q' \) such that \( q < q' \), then the local consistency of Definition 7 will be violated because \( \emptyset \cup (Q_1^1 \setminus C^0) \models \delta(q', w[j - 1]) \). If \( q' \in (C^i \cap Q_1^{i-1}) \setminus D^{j-1} \), by definition, \( q \preceq_{j-1} q' \) and thus also \( q \preceq_{j-1} q' \) since \( q' \) is a maximal element w.r.t. \( \preceq_{j-1} \). Moreover, by R1’, there exists an \( r \)-state for \( q \) and \( q' \) satisfying Definition 7. If \( d_{j-1}(q') \leq i - (j - 1) = 1 \), i.e., \( d_{j-1}(q') = 1 = d_{j-1}(q) \), this violates the existence of \( r \)-state in Definition 7, according to Definition 5. Thus, \( d_{j-1}(q') > i - (j - 1) = 1 \). It follows that Item (2) holds. Now we only need to prove that \( d_{j-1}(q) \leq d_{j-1}(q') \Leftrightarrow q \preceq_{j-1} q' \) and \( d_{j-1}(q) \leq i - (j - 1) \) hold when \( q' \in (C^i \cap Q_1^{i-1}) \setminus D^{j-1} \). The case when \( q' \in D^{j-1} \) has already been proved above.

By Item (2), we already have \( d_{j-1}(q) < d_{j-1}(q), q \preceq_{j-1} q' \) and clearly, \( d_{j-1}(q) \leq 1 \) hold.

Hence, Item (1) holds as well. It follows that when \( j = i \), the three items also hold when \( j \rightarrow j - 1 \). If there are no such states, i.e., \( D^{j-1} = \emptyset \), then the backwards deterministic definition of \( (Q_1^{i-1}, \preceq_{j-1}) \) from \( (Q_1^1, \preceq_j) \) according to Rule R1’ implies (with the absence states \( d_{j-1}(q) = 1 \) and Rule R1) that all states in \( C^i \cap Q_1^{i-1} \) are identified by \( \preceq_{j-1} \) and \( D^{j-1} = \emptyset \). Such a macrostate is accepting, which contradicts \( j > j' + 1 \).

For \( j < i \), we first observe that, for all states \( q \in C^i \cap Q_1^{i-1}, d_{j-1}(q) \leq i - j + 1 \) holds iff \( q \in D^{i-1} \) with the same backwards deterministic argument as above (using Rules R1

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6 note that \( D^j = \emptyset \), so that all states in \( C^i \cap Q^j \) are maximal elements of, and therefore identified by \( \preceq_j \)
and $R1′$) from the induction hypothesis. But we also have to establish that there is a state $q ∈ D_j^{j+1} ⊆ C^i ∩ Q^j_1$ with $d_{j-1}(q) = i − j + 1$.

We assume for contradiction that this is not the case. Then $\sup\{d_{j-1}(q) \mid q ∈ D_j^{j-1}\} ≤ i − j$, which implies that the set $\widehat{D}_j = \{q ∈ C^i ∩ Q^j_1 \mid d_j(q) < i − j\}$ is a downwards closed (w.r.t. $≤_j$) set, which is strictly smaller than $D_j$ and satisfies the other transition requirements. Therefore $D_j$ does not satisfy the minimality requirement (contradiction).

The proof for the three items are then easy. First, Item (3) has been proved above. By induction hypothesis, we have that the three items hold on position $j$. We now prove Item (2). For all $q ∈ C^i ∩ Q^j_1$ and $q′ ∈ M_j^{j-1} = (C^i ∩ Q^j_1) \setminus D_j^{j-1}$ (if it exists), by definition $q′$ is a maximal element w.r.t. $≤_{j-1}$. Clearly, $q ≤_{j-1} q′$ and thus $q ≤_{j-1} q′$. Suppose $d_j(q′) ≤ i − j + 1$. But then we have a state $q ∈ D_j^{j-1}$ such that $d_j(q) = i − j + 1$. Since $q ≤_{j-1} q′$, there must exist an $r$-state in $C^i \cap Q^j_1$ satisfying $R1′$ of Definition 7. By Definition 5, $d_j(r) ≤ i − j$. We then have that $r ∈ D_j$ because there is a state $r′ ∈ D_j$ such that $d_j(r′) = i − j$ and then we have $r ≤ j r′$ by Item (1). (If $D_j = ∅$, it immediately leads to contradiction.) But then, $\{p ∈ C^i \cap Q^j_1 \mid p ≤_{j-1} r\} \cup (Q^j_1 \setminus C^j) ≳ (q,w[j − 1])$ since $\{p ∈ C^i \cap Q^j_1 \mid p ≤_{j-1} r\} = \{p ∈ C^i \cap Q^j_1 \mid d_j(p) ≤ d_j(r) − 1 < i − j = d_{j-1}(q) − 1\}$ (by induction hypothesis), violating Definition 5. It follows that $d_{j-1}(q′) > i − j + 1$. Therefore, Item (2) holds. Item (1) can be proven similarly. One can also prove similarly the following when $C^i ⊆ A$:

1. for all $q ∈ D_j$, and all $q′ ∈ C^i \cap Q^j_2$, $d_j(q′) ≤ d_j(q′) ⇔ q ≤_{j} q′$, and $d_j(q) ≤ i − j$ hold,
2. for all $q ∈ C^i \cap Q^j_2$ and all $q′ ∈ (C^i \cap Q^j_2) \setminus D_j$. $q ≤_{j} q′$ and $d_j(q′) > i − j$ hold, and
3. $m_j = \sup\{d_j(q) \mid q ∈ D_j\} = i − j$, using $\sup\emptyset = 0$.

This closes the inductive argument.

Finally, we observe that the simulation macrorun $\widehat{ρ}$ for the sole accepting macrorun of $B_\alpha$ is the only macrorun that satisfies the three item requirements, based on following facts. (i) There is only a single initial macrostate $(Q^0_1,Q^0_2,≤_0,C^0,D^0)$ that fits $R_\alpha$ (with all states in $C^0 ∩ Q^0_1$ or $(C^0 ∩ Q^0_2)$ being maximal w.r.t. $≤_0$ since $D^0 = ∅$), and when we take a transition from an accepting macrostate $(Q^j_1,Q^j_2,≤_j,C^j,D^j = ∅)$ (including the first), the next SCC $C^{j+1} = \text{next}(C^j)$ is deterministically selected. (ii) Moreover, all relevant states from the SCC $C^{j+1} \cap Q^j_1$ (resp. $C^{j+1} \cap Q^j_2$) are in the $D^{j+1}$ component, since $D^{j+1} = C^{j+1} ∩ Q^j_1$ (resp. $D^{j+1} = C^{j+1} ∩ Q^j_2$) by construction. We have seen that $\sup\{d_j(q) \mid q ∈ C^j \cap Q^j_1\}$ (resp. $\sup\{d_j(q) \mid q ∈ C^j \cap Q^j_2\}$) determines the distance to the next breakpoint, by Item (3) when $D^{j+1} = ∅$ for all $j > 0$, and thus the $≤_j$ and $D^j$ up to the next breakpoint. Wrong guesses of the preorders for the states in $D$-component and the states in $M$ will lead to violation to $R1′$ and $R2′$ in the local consistency test. With a simple inductive argument we can thus conclude that there can only be one such accepting macrorun. □